PHYS 942 homework assignment #03

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PHYS 942
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Names ($\leq$3, write clearly): ______________________________

Due: Friday, October 12, at the lecture. **Show all your steps!**

1. (*40 points*) The dispersion relation of the magnetohydrodynamic (MHD) equations (see Jackson 7.7 or any decent MHD book) yields three different waves with their phase velocities given by:

$$V_1^2 = (v_A \cos\theta)^2$$

and

$$V_{2,3}^2 = \frac{1}{2}(c_s^2 + v_A^2) \pm \frac{1}{2}\sqrt{(c_s^2 + v_A^2)^2 - 4c_s^2 v_A^2 \cos^2\theta}$$

where $c_s = \sqrt{\gamma p_0/\rho_0}$ is the sound speed, $v_A = \sqrt{B_0^2/\mu_0\rho_0}$ is the Alfvén speed, and p_0, ρ_0, B_0, and γ are the plasma pressure, density, magnetic field, and ratio of specific heats, respectively. The field B_0 lies along the z-axis, and θ is the angle between $\mathbf{k}$ and the field $\mathbf{B}_0$. The wave with phase speed V_1 is called the Alfvén wave. The other two are called the fast and slow magnetosonic waves, or fast-mode / slow-mode for short.

In the case of MHD waves k does not depend on ω, but it strongly depends on the propagation direction θ, and our definition of the group speed $\mathbf{v}_G = \nabla_{\mathbf{k}}\omega(\mathbf{k})$ still applies. Assume that c_s and v_A are given.

(a) Calculate the group velocity of the Alfvén wave and make a polar plot of V_1 and $v_G(\theta)$.

(b) Calculate the group velocity of the fast mode wave and make polar plots of V_2 and $v_G(\theta)$ for $c_s = 1,\ v_A = 2$ and for $c_s = 1,\ v_A = 0.5$.

(c) For extra credit (20 points) calculate the group velocity of the slow mode wave and make polar plots of V_3 and $v_G(\theta)$ for the same parameter combinations as above. This is a bit difficult because the slow mode wave only propagates for a limited range of θ values.

Hint: You need to write the wave speeds in terms of k_x and k_z first and then go back to spherical coordinates. Remark: The MHD waves are a good example of how different the phase and the group speed can be. The polar diagrams are called the *Fridrichs-I* and *Fridrichs-II* diagrams for the phase and group speeds, respectively.

2. (*40 points*) Consider a circularly polarized plane wave propagating in a homogeneous medium in the z-direction. The wave has a *finite extent* in x and y given by $E_0(x, y)$, such that the lateral extent of the wave is large compared to its wavelength and that $E_0(x, y)$ is slowly varying. Such a wave could be produced, for example, by a laserpointer.

   Show that

   (a) the electric field of the wave is given by

   $$\mathbf{E}(x, y, z, t) = \Big[E_0(x, y)(\mathbf{e}_x \pm i\mathbf{e}_y) + \frac{i}{k}(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial x})\mathbf{e}_z\Big]e^{ikz-i\omega t}.$$

   (b) the magnetic field is approximately given by

   $$\mathbf{B} \simeq \pm i\sqrt{\mu\epsilon}\,\mathbf{E}.$$

   (c) assuming that E_0 is real, the ratio of the z-component of the angular momentum density ($\mathbf{L} = \epsilon\mathbf{x}\times(\mathbf{E}\times\mathbf{B})$) to the energy density is

   $$\frac{L_z}{U} = \pm\omega^{-1}.$$

   What does this imply for the quantization of photons?

3. (*20 points*) The properties of the ionosphere with respect to EM wave propagation is well described by the plasma dispersion in the high frequency limit $\epsilon(\omega)/\epsilon_0 = 1 - \omega_p^2/\omega^2$, where ω_p is the plasma frequency.

   Consider an ionosphere layer where ω_p jumps suddenly from zero to a given value at a height h.

   (a) For waves with polarization both perpendicular to the plane of incidence and parallel to the plane of incidence, show from the Fresnel equations that for $\omega > \omega_p$ there is a range of angles of incidence for which reflection is not total, but for larger angles there is total reflection back towards earth.

   (b) A radio amateur operating at a wavelength of 30 meters during the day finds that she can receive distant stations located more than 1000 km away, but none closer. Assuming that the signals are being reflected at the ionosphere F layer at 300 km height, calculate the electron density.

   (c) Explain why late at night she can no longer receive any station more than 1000 km away.

   (d) Explain why AM stations (0.5 to 1.5 MHz) can be received closeby ($<$100 km) but typically not farther away.

   (e) Explain why FM stations (80 to 110 MHz) can be received closeby ($<$100 km) but typically not farther away.

(f) Explain why short wave (SW) stations can be received halfway around the globe, say from 10,000 km away.

(g) The Russian Sputnik I satellite transmitted its (in)famous “beep – beep – beep” at frequencies of 20 and 40 MHz. Explain why was this a useful scientific experiment. And why did the beeps have an enourmous propaganda value?