Lec23 IAM550 J. Raeder 11/19/2019 Differential equations, Linear systems

Announcements:

- Final: TBD

All on the blackboard or real time MATLAB

Recap: ODE, Euler forward method

Demonstrate: Euler forward instability

Better methods:

- Euler backward → stable, but implicit
- Predictor - Corrector → similar stability, but more accurate, show graphically

$$y' = f(t,y), \quad y(t_0) = y_0$$

do one Euler forward step:

$$\tilde{y}_{i+1} = y_i + hf(t_i, y_i)$$

and average with an Euler backward step, using the predicted values:

$$y_{i+1} = y_i + \tfrac{1}{2}h\big(f(t_i, y_i) + f(t_{i+1}, \tilde{y}_{i+1})\big).$$

Alternatively, make a half forward step, then evaluate slope at predicted point (second order Runge-Kutta):

$$y_{n+1} = y_n + hf\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(t_n,\ y_n)\right).$$

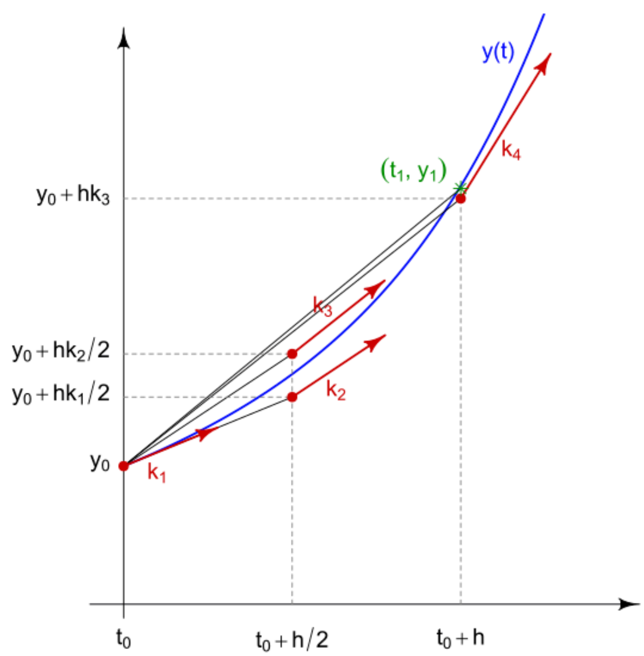
→ explain on bb.

Most popular: 4th order Runge-Kutta, a.k.a. RK4 (for reference):

$$\begin{aligned} k_1 &= h\ f(t_n, y_n), \\ k_2 &= h\ f\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right), \\ k_3 &= h\ f\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right), \\ k_4 &= h\ f\left(t_n + h, y_n + k_3\right). \end{aligned}$$

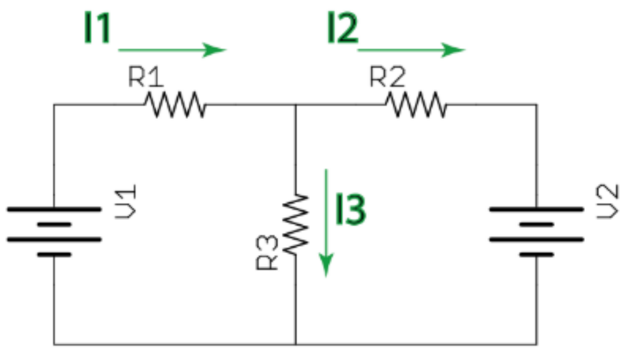
$$\begin{aligned} y_{n+1} &= y_n + \tfrac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), \\ t_{n+1} &= t_n + h \end{aligned}$$

→ very efficient and accurate, easy to program, only needs 4 function evaluations for every step.



Systems of linear equations:

Example: resistivity network:



Using Kirchhoff' s laws:

1. The sum of voltages around any loop is zero.
2. The sum of currents at any junction is zero.

Applying the voltage law to the left-hand loop, we get

$$V_1 - I_1 R_1 - I_3 R_3 = 0$$

From the right-hand loop, we get

$$V_2 + I_2 R_2 - I_3 R_3 = 0$$

We need one more equation, for which we can use the junction at top center and the current law:

$$I_1 - I_2 - I_3 = 0$$

Can be written as a system of linear equations for the currents:

$$\begin{aligned} R_1 I_1 + 0 I_3 + R_3 I_3 &= \quad V_1 \\ 0 I_1 - R_2 I_2 + R_3 I_3 &= \quad V_2 \\ I_1 - I_2 - I_3 &= \quad 0 \end{aligned}$$

Or more compact in matrix form:

$$\begin{bmatrix} R_1 & 0 & R_3 \\ 0 & -R_2 & R_3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ 0 \end{bmatrix}$$

Solution procedure: Gauss Elimination

Basic rules: The following does not change the solution:

1. Multiplying any equation with a constant
2. Adding 2 equations
3. Swapping equations

Basic idea:

1. Elimination: make the lower triangle below the diagonal zero by multiplying and adding. Column by column downward, starting from first column.
2. Back substitution: Now solve for equations from bottom to top.

Example. Do on the board, but only w/matrix. Also, avoid fractions.

System of equations	Row operations	Augmented matrix
$\begin{aligned} 2x+y-z&=8 \\ -3x-y+2z&=-11 \\ -2x+y+2z&=-3 \end{aligned}$		$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right]$
$\begin{aligned} 2x+y-z&=8 \\ \tfrac{1}{2}y+\tfrac{1}{2}z&=1 \\ 2y+z&=5 \end{aligned}$	$L_2+\tfrac{3}{2}L_1 \to L_2$ $L_3+L_1 \to L_3$	$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ 0 & \tfrac{1}{2} & \tfrac{1}{2} & 1 \\ 0 & 2 & 1 & 5 \end{array}\right]$
$\begin{aligned} 2x+y-z&=8 \\ \tfrac{1}{2}y+\tfrac{1}{2}z&=1 \\ -z&=1 \end{aligned}$	$L_3+-4L_2 \to L_3$	$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ 0 & \tfrac{1}{2} & \tfrac{1}{2} & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$
The matrix is now in echelon form (also called triangular form)		
$\begin{aligned} 2x+y&=7 \\ \tfrac{1}{2}y&=\tfrac{3}{2} \\ -z&=1 \end{aligned}$	$L_2+\tfrac{1}{2}L_3 \to L_2$ $L_1-L_3 \to L_1$	$\left[\begin{array}{ccc\|c} 2 & 1 & 0 & 7 \\ 0 & \tfrac{1}{2} & 0 & \tfrac{3}{2} \\ 0 & 0 & -1 & 1 \end{array}\right]$
$\begin{aligned} 2x+y&=7 \\ y&=3 \\ z&=-1 \end{aligned}$	$2L_2 \to L_2$ $-L_3 \to L_3$	$\left[\begin{array}{ccc\|c} 2 & 1 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$
$\begin{aligned} x&=2 \\ y&=3 \\ z&=-1 \end{aligned}$	$L_1-L_2 \to L_1$ $\tfrac{1}{2}L_1 \to L_1$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$

Code: next time