Lec21 IAM550 J. Raeder 11/12/2019 Exponential growth, Differential equations

Announcements:

- Midterm II grades by the end of the week.
- Final: in-class or take-home (8h: 9-5 on 12/17) → think about it

All on the blackboard or real time MATLAB

Exponential growth, a.k.a. geometrical growth:
Something grows (or declines) by a certain fraction/percentage over same time interval:

$$x_t = x_0(1+r)^t$$

examples:

- bacterial infections (sepsis) → doubles every 20 min.
- nuclear chain reaction → every neutron produces 2 new neutrons every 2 microseconds.
- Nuclear decay → number of particles decaying is proportional to number of particles remaining (negative *r*). C^{14} dating.
- Capacitor discharge → discharge current is proportional to remaining charge.
- Compound interest → growing by percentage.
- Ponzi scheme → every investor must find N new investors.
- Rice on a chessboard → 2^64 ~ 1 trillion on the 41st square!
- Human population → Club of Rome, 'The Limits to Growth' Dennis Meadows
- And many more

Compared to other growth:

$$\lim_{t\to\infty} \frac{t^\alpha}{ae^t} = 0.$$

The exponential function grows faster than any power!

Mathematically:

$$x(t) = a \cdot b^{t/\tau} \qquad x(0) = a$$

tau is a time constant:

if b=e → tau is the e-folding time
if b=2 → tau is the doubling time (half-life if tau is negative)

Can be seen here:

$$x(t+\tau) = a \cdot b^{\frac{t+\tau}{\tau}} = a \cdot b^{\frac{t}{\tau}} \cdot b^{\frac{\tau}{\tau}} = x(t) \cdot b \, .$$

One can always use *b=e* by adjusting tau.

The essence of exponential growth is that the rate of change (slope, first derivative) of x(t) is proportional to the value of x(t). This leads to a differential equation:

$$\frac{dx}{dt} = kx$$

The differential equation is solved by direct integration:

$$\frac{dx}{dt} = kx$$

$$\frac{dx}{x} = k\,dt$$

$$\int_{x(0)}^{x(t)} \frac{dx}{x} = k \int_0^t \, dt$$

$$\ln \frac{x(t)}{x(0)} = kt.$$

so that

$$x(t) = x(0)e^{kt}$$

Mathematically this is an **Ordinary Differential Equation (ODE)** and an ***Initial Value Problem (IVP).***

ODEs have only one independent variable, as opposed to ***PDEs (Partial Differential Equations).***

Other IVPs that you know from physics:

Equations of motion:

$d^2x/dt^2 = F/m$ → can be written as 2 first order differential equations:

$dv/dt = F/m$ and $dx/dt = v$ → all higher order differential equations can be written as a system of first order differential equations → we only need to know how to solve first order differential equations ☺

Depending on F there may or may not be a analytic solution.

Pendulum equation:

Strictly:

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0 \qquad \textbf{Eq. 1}$$

for small angle:

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0.$$

which has an analytic solution:

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{\ell}}\,t\right)$$

What if is θ is not small and we need to solve Eq. 1?

Numerical solution, the most simple way:

Take finite steps in time: $t_0, t_1, t_2, \cdots.\ t_n = n*h$

For a general differential equation:

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0.$$

Use a Taylor series expansion

$$y(t_0 + h) = y(t_0) + hy'(t_0) + \frac{1}{2}h^2 y''(t_0) + O(h^3).$$

to get:

$$y_{n+1} = y_n + hf(t_n, y_n).$$

with the local truncation error (LTE)

$$\mathrm{LTE} = y(t_0 + h) - y_1 = \frac{1}{2}h^2 y''(t_0) + O(h^3).$$

→ MATLAB code on the fly, compare to exact, show instability

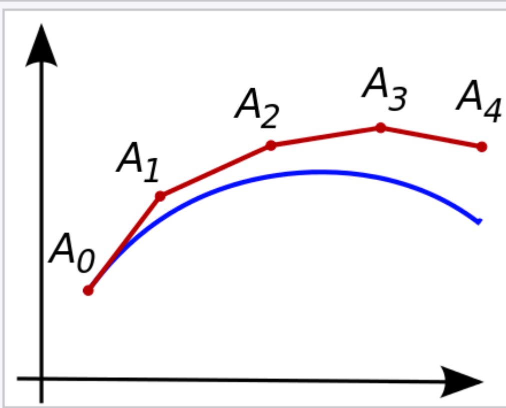

Illustration of the Euler method. The unknown curve is in blue, and its polygonal approximation is in red.