Numerical Integration (Quadrature)

IAM 550, Lec14, 2019-10-10, J. Raeder

From Wikipedia: Riemann was the second of six children, shy and suffering from numerous nervous breakdowns. Riemann exhibited exceptional mathematical skills, such as calculation abilities, from an early age but suffered from timidity and a fear of speaking in public.

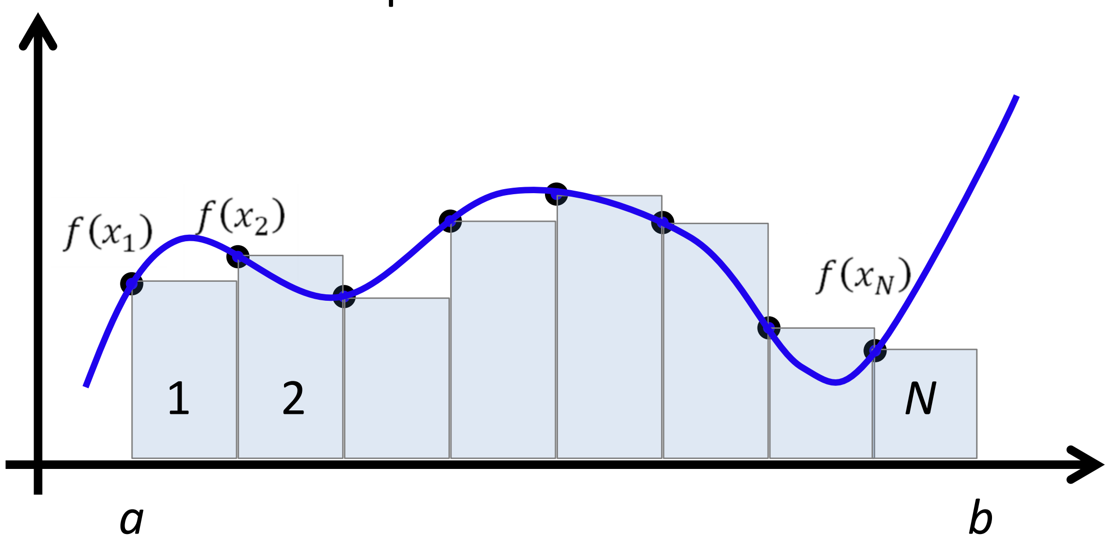
Bernhard Riemann, 1826-1866 (You should have met him in Calc II)

Announcements

- **Midterm exam:**
  - **17 Oct, 2:10 – 3:20 (70 minutes)**
  - **N108 (A-O) and W114 (P-Z)**
  - **115 points total, points over 100 carry over to final**
  - **Material up to and including Lecture 13**
  - **Students with SAS letter come see me**
- **Homework 2 will be on the web page by Friday COB. Due 2 weeks hence.**

Numerical Integration

Left endpoint

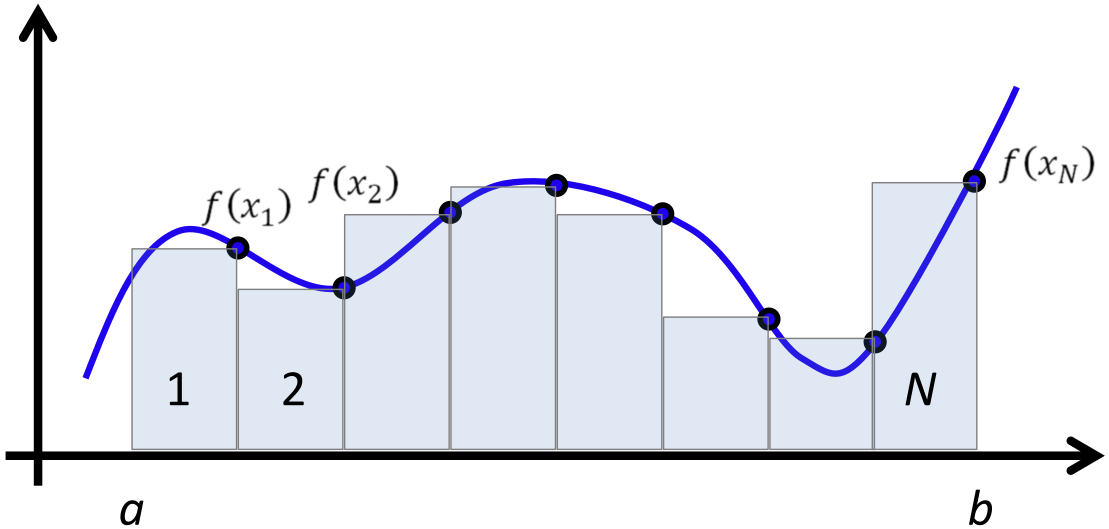


$$\int_a^b f(x)dx \cong \sum_{n=1}^{N} f(x_n)dx$$

A sum of rectangles

Numerical Integration

Right endpoint

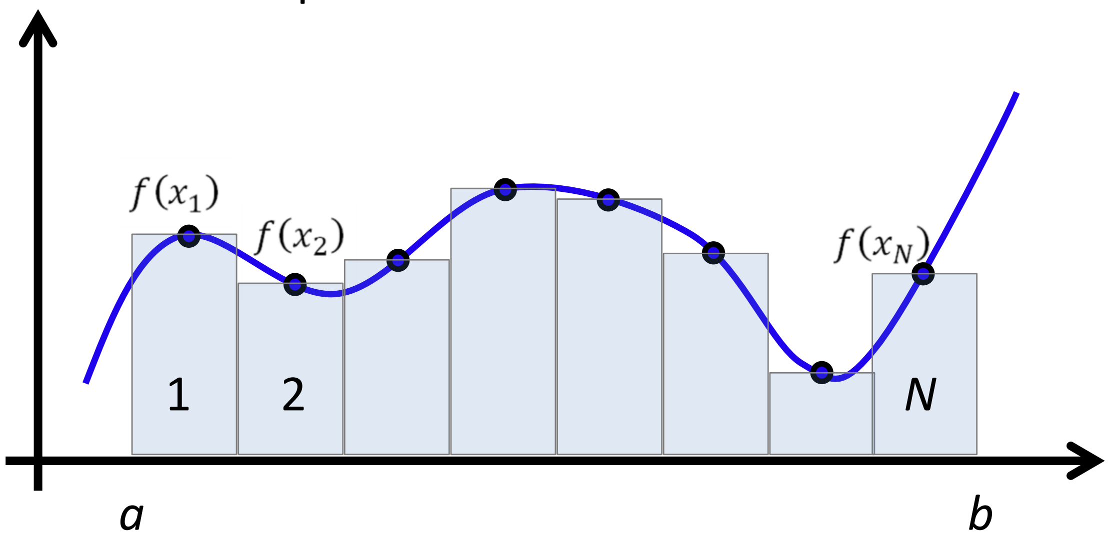


$$\int_a^b f(x)dx \cong \sum_{n=1}^{N} f(x_n)dx$$

A sum of rectangles

Numerical Integration

Midpoint



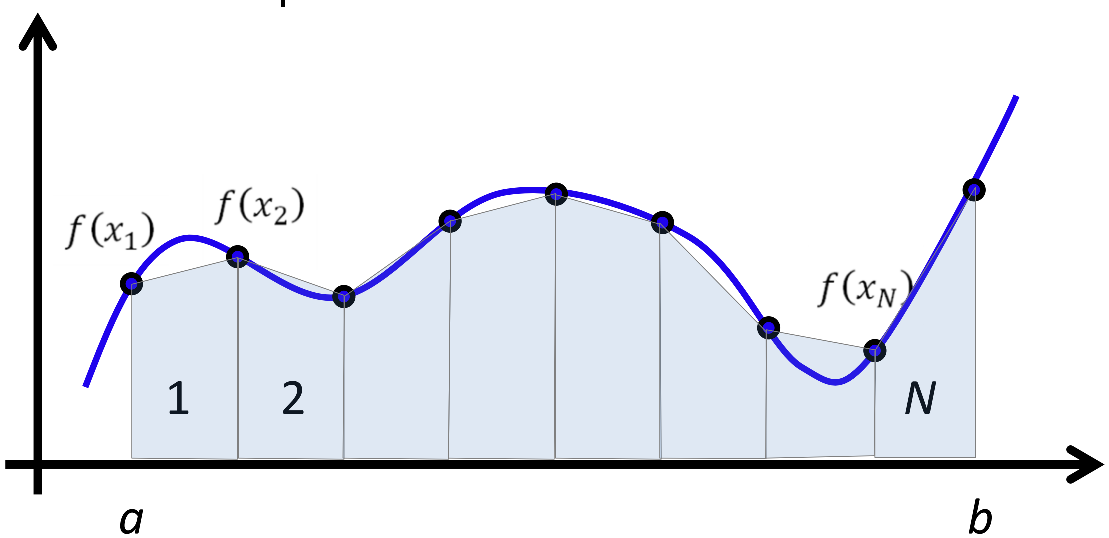
$$\int_a^b f(x)dx \cong \sum_{n=1}^{N} f(x_n)dx$$

A sum of rectangles

The simple Riemann sum. Will converge for N → Inf, but maybe very slowly.

Numerical Integration

Trapezoid

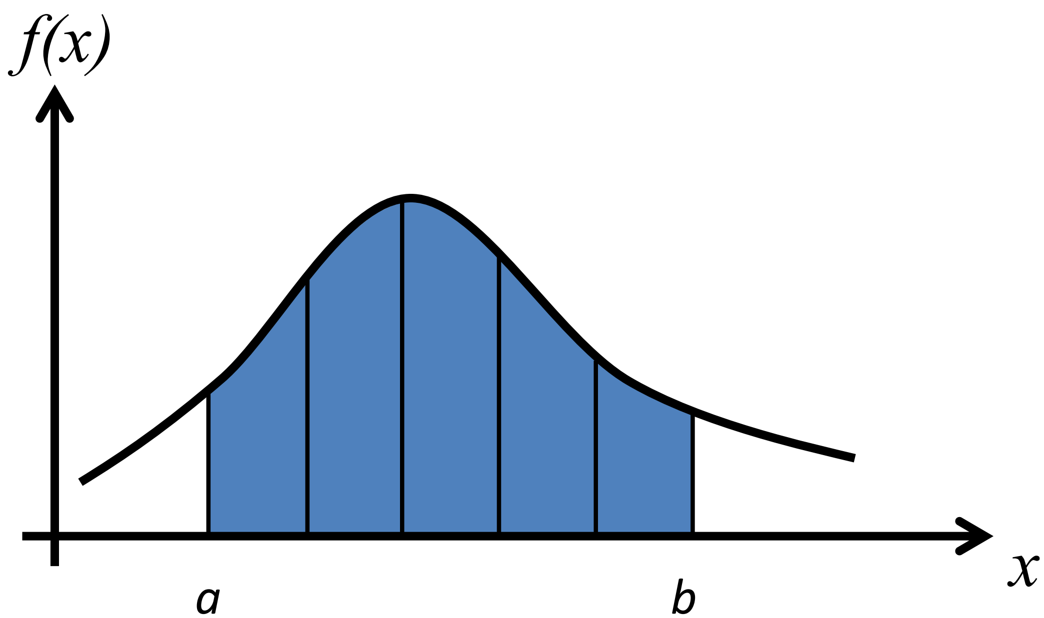


Probably a little better.

A more formal introduction to numerical integration

We want to approximate the definite integral:

$$\int_a^b f(x)dx$$



1. Divide the range of integration into n-1 equal intervals of length: $$\frac{b-a}{n-1}$$

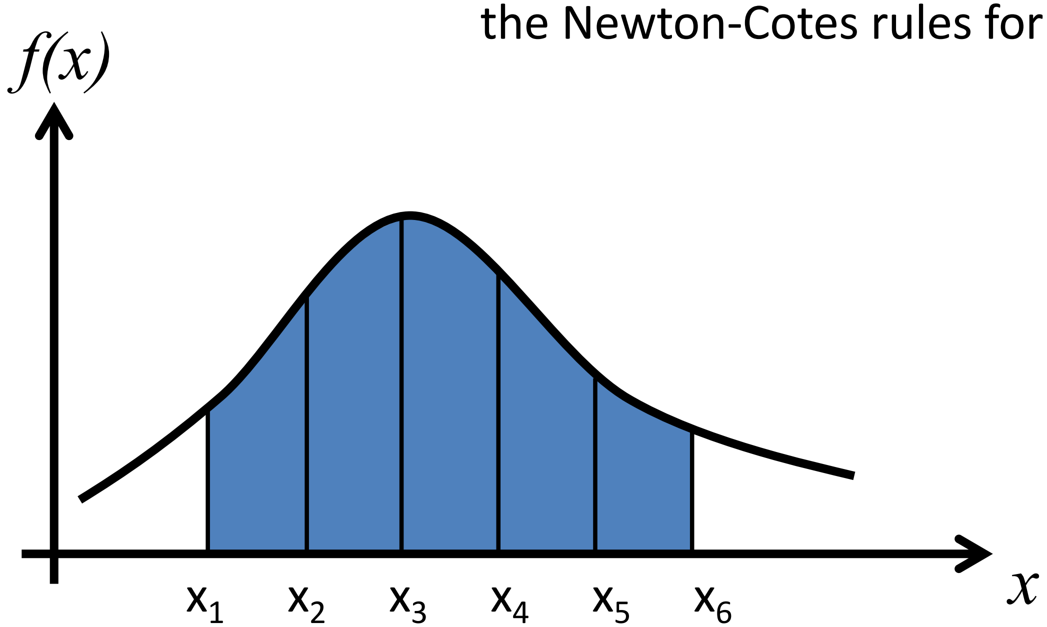
2. Replace $f(x)$ with an n^{th} order polynomial that is easy to integrate: $$f(x) \cong f_n(x) = a_o + a_1 x + a_2 x^2 + \cdots a_n x^n$$

A more formal introduction to numerical integration

2. Replace *f(x)* with an n^{th} order polynomial that is easy to integrate:

$$f(x) \cong f_n(x) = a_o + a_1 x + a_2 x^2 + \cdots a_n x^n$$

If we make sure that these two equal each other at the points x_i then we are following the Newton-Cotes rules for quadrature.

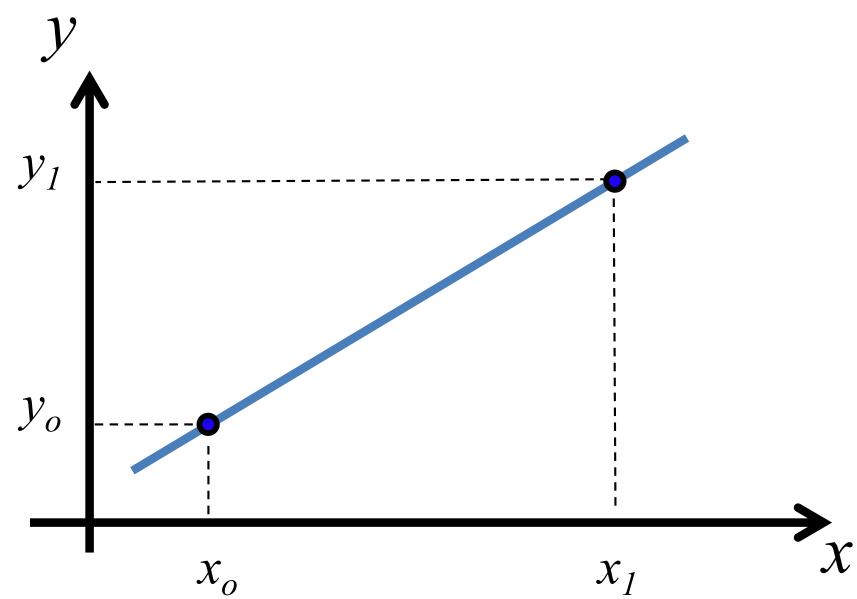


Lagrange Approximation

$$f(x) \cong f_n(x)$$

If we make sure that these two equal each other at the points x_i then we are following the Newton-Cotes rules for quadrature.

How do we do this?



What we learned way back when…

$$y = y_o + m(x - x_o)$$

$$m = \frac{(y_1 - y_o)}{(x_1 - x_o)}$$

$$y = P(x) = y_o + (y_1 - y_o)\frac{(x - x_o)}{(x_1 - x_o)}$$

Or as Lagrange would have written:

$$y = P_1(x) = y_o\frac{(x - x_1)}{(x_o - x_1)} + y_1\frac{(x - x_o)}{(x_1 - x_o)}$$

Lagrange Approximation

Or as Lagrange would have written:

$$y = P_1(x) = y_o \frac{(x - x_1)}{(x_o - x_1)} + y_1 \frac{(x - x_o)}{(x_1 - x_o)}$$

$$L_{1,0}(x) = \frac{(x - x_1)}{(x_o - x_1)} \qquad L_{1,1}(x) = \frac{(x - x_o)}{(x_1 - x_o)}$$

Lagrange coefficient polynomials

So we can write in a type of shorthand:

$$P_1(x) = \sum_{k=0}^{1} y_k L_{1,k}(x)$$

Lagrange Approximation

For an N^{th} degree polynomial approximation that is exact at the points x_i:

$$P_N(x) = \sum_{k=0}^{N} y_k L_{N,k}(x)$$

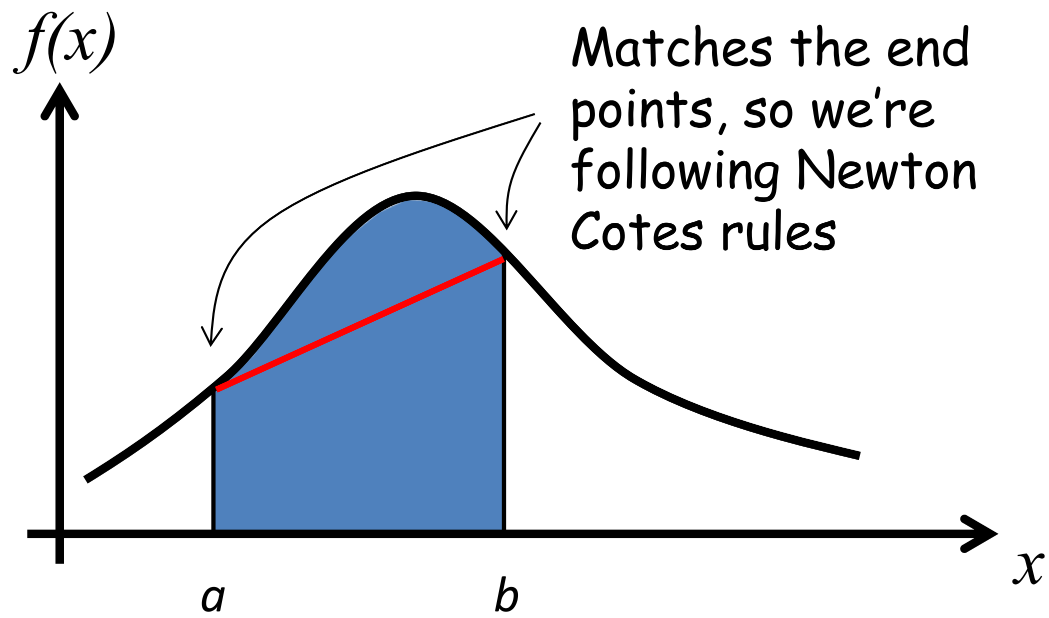
$$L_{N,k}(x) = \frac{(x - x_o)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_N)}{(x_k - x_o)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_N)}$$

Note that $(x-x_k)$ and (x_k-x_k) are not present here.

So lets try it:

We want to approximate the definite integral:

$$\int_a^b f(x)dx$$



1. Divide the range of integration into n-1 equal intervals of length: $\dfrac{b-a}{n-1}$

Note that n = 2 in this example.

2. Replace *f(x)* with an n^{th} order polynomial that is easy to integrate: $P_1(x) = \sum_{k=0}^{1} y_k L_{1,k}(x)$

$$L_{1,0}(x) = \frac{(x - x_1)}{(x_o - x_1)}$$

$$L_{1,1}(x) = \frac{(x - x_o)}{(x_1 - x_o)}$$

So lets try it:

We want to approximate the definite integral:

$$\int_a^b f(x)dx$$

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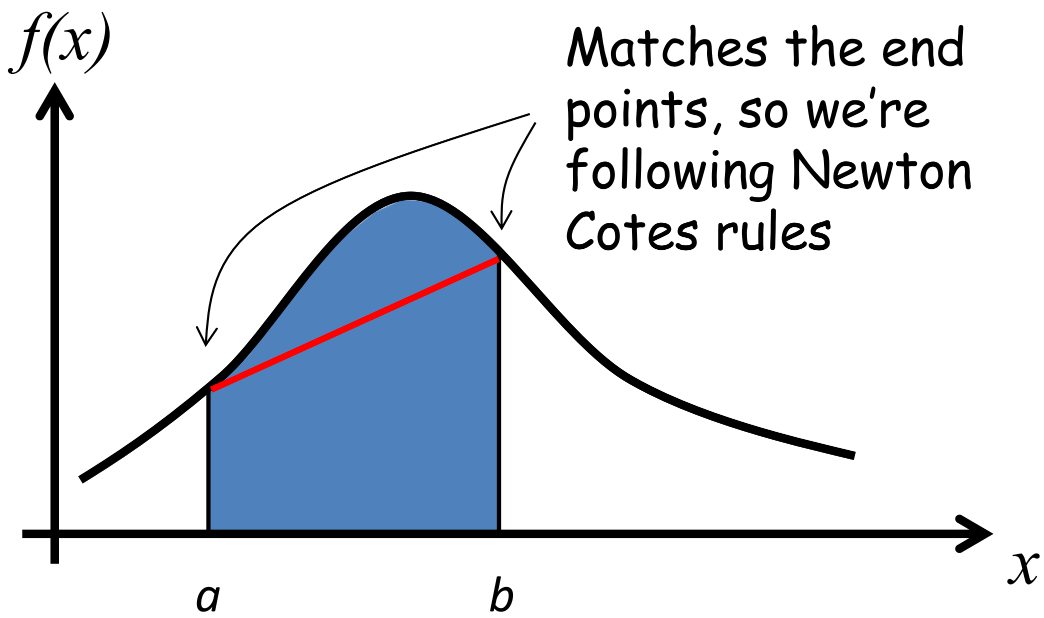


1st term

$$y(a)\int_a^b \frac{(x-b)}{(a-b)}dx = y(a)\frac{1}{2(a-b)}(x-b)^2\Big]_a^b = y(a)\frac{(b-a)}{2}$$

2nd term

$$y(b)\int_a^b \frac{(x-a)}{(b-a)}dx = y(b)\frac{1}{2(b-a)}(x-a)^2\Big]_a^b = y(b)\frac{(b-a)}{2}$$

So lets try it:

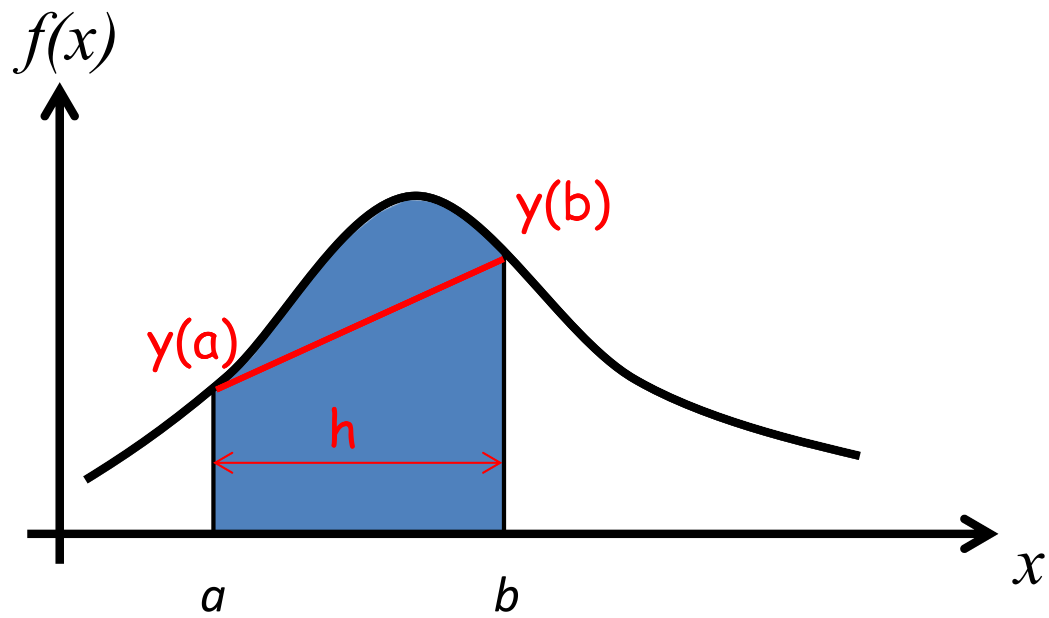
We want to approximate the definite integral:

$$\int_a^b f(x)dx$$

$$P_1(x) = \sum_{k=0}^{1} y_k L_{1,k}(x)$$

$$L_{1,0}(x) = \frac{(x - x_1)}{(x_o - x_1)}$$

$$L_{1,1}(x) = \frac{(x - x_o)}{(x_1 - x_o)}$$



This is a sum, so we add these two:

$$y(a)\frac{(b-a)}{2} + y(b)\frac{(b-a)}{2} = \frac{h}{2}[y(a) + y(b)]$$

This is the area of the trapezoid above.

This is the Newton-Cotes Formula for n = 2
a.k.a trapezoidal rule.

Error in the Trapezoidal Rule

We want to approximate the definite integral:

$$\int_a^b f(x)dx$$

$$P_1(x) = \sum_{k=0}^{1} y_k L_{1,k}(x)$$

$$L_{1,0}(x) = \frac{(x - x_1)}{(x_o - x_1)}$$

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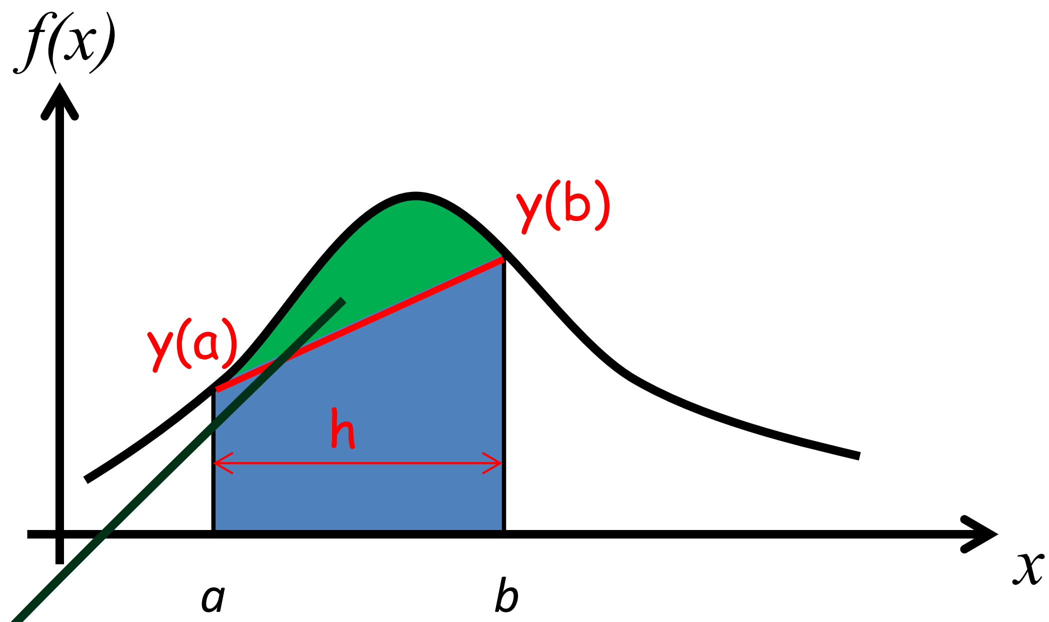

$$E = \int_a^b f(x)dx - \frac{\mathrm{h}}{2}[\mathrm{y(a)} + \mathrm{y(b)}]$$

Error in the Trapezoidal Rule

We want to approximate the definite integral:

$$\int_a^b f(x)dx$$

$$P_1(x) = \sum_{k=0}^{1} y_k L_{1,k}(x)$$

$$L_{1,0}(x) = \frac{(x - x_1)}{(x_o - x_1)}$$

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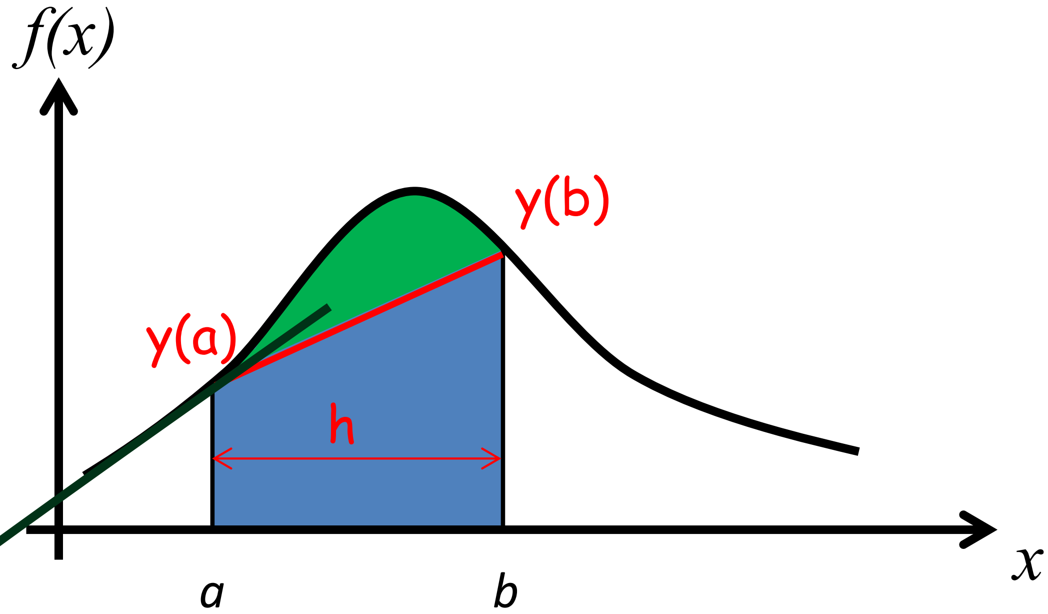


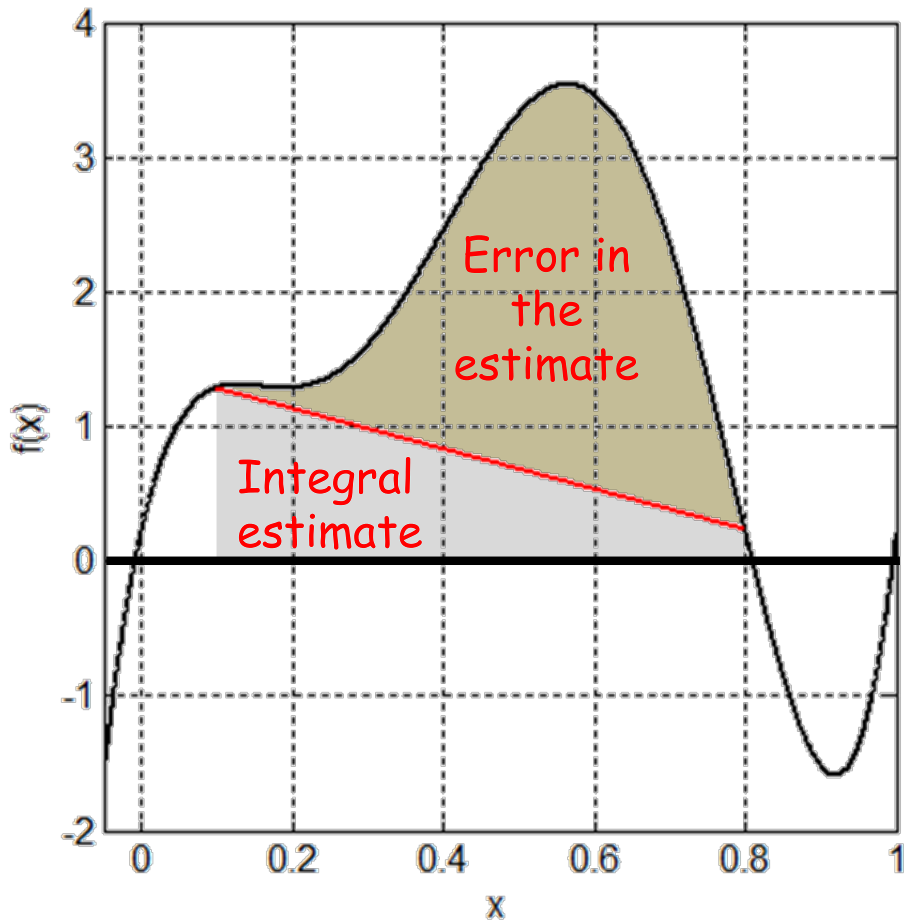
$$E = \int_a^b f(x)dx - \frac{h}{2}[y(a) + y(b)] = -\frac{1}{12} f''(\xi)\,(\mathrm{b} - \mathrm{a})^3$$

Note that if f(x) is a line, there is no error!

Example

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

Integrate *f(x)* from a = 0.1 to b=0.8



True answer: 1.5471

f(0.1) = 1.289
f(0.8) = 0.232
h = 0.7

Trapezoid rule estimate:
h/2*[f(0.1) + f(0.8)] = 0.5324

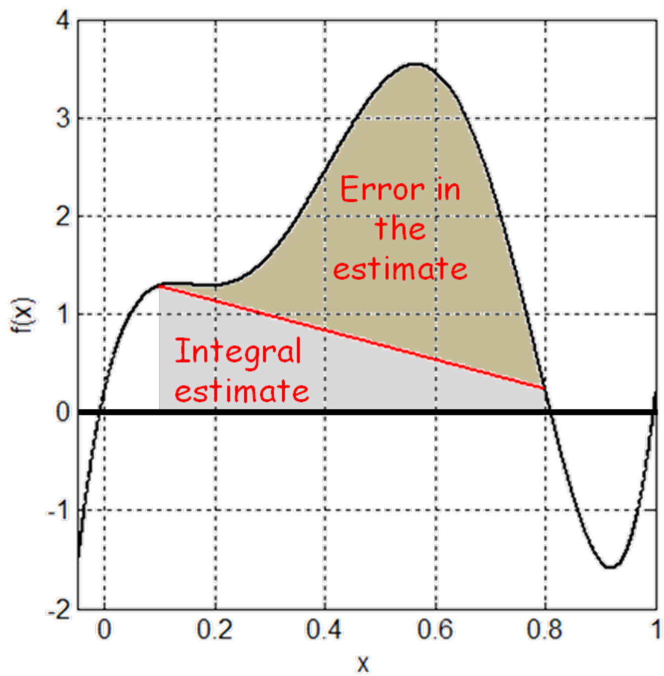
E = 1.5471-0.5324 = 1.015
Percent error = 65.6%

Note that we don't always know this error!

Example

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

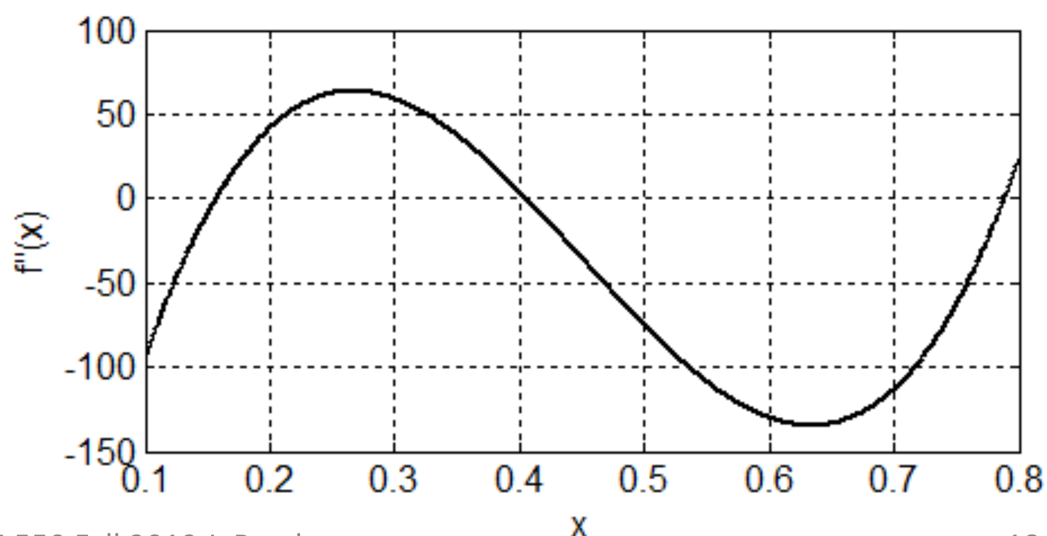
Integrate *f(x)* from a = 0.1 to b=0.8



A close look at the error

$$E = \int_a^b f(x)dx - \frac{h}{2}[y(a) + y(b)] = -\frac{1}{12} f''(\xi)\,(\mathrm{b}-\mathrm{a})^3$$

$$f''(x) = -400 + 4{,}050x - 10{,}800x^2 + 8{,}000x^3$$

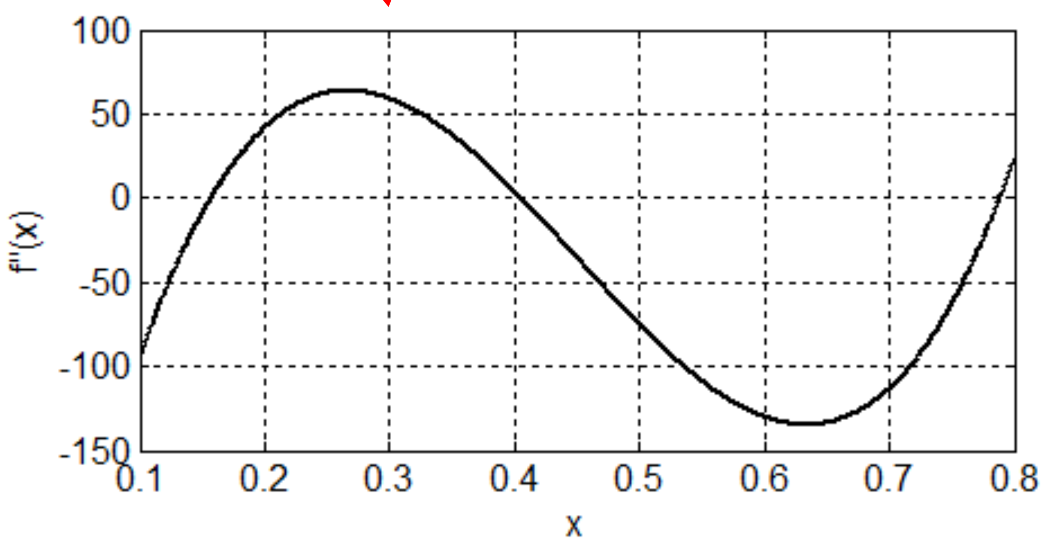


Example

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

$$E = -\frac{1}{12} f''(\xi)\,(\mathrm{b}-\mathrm{a})^3$$

$0.7^3 = 0.343$

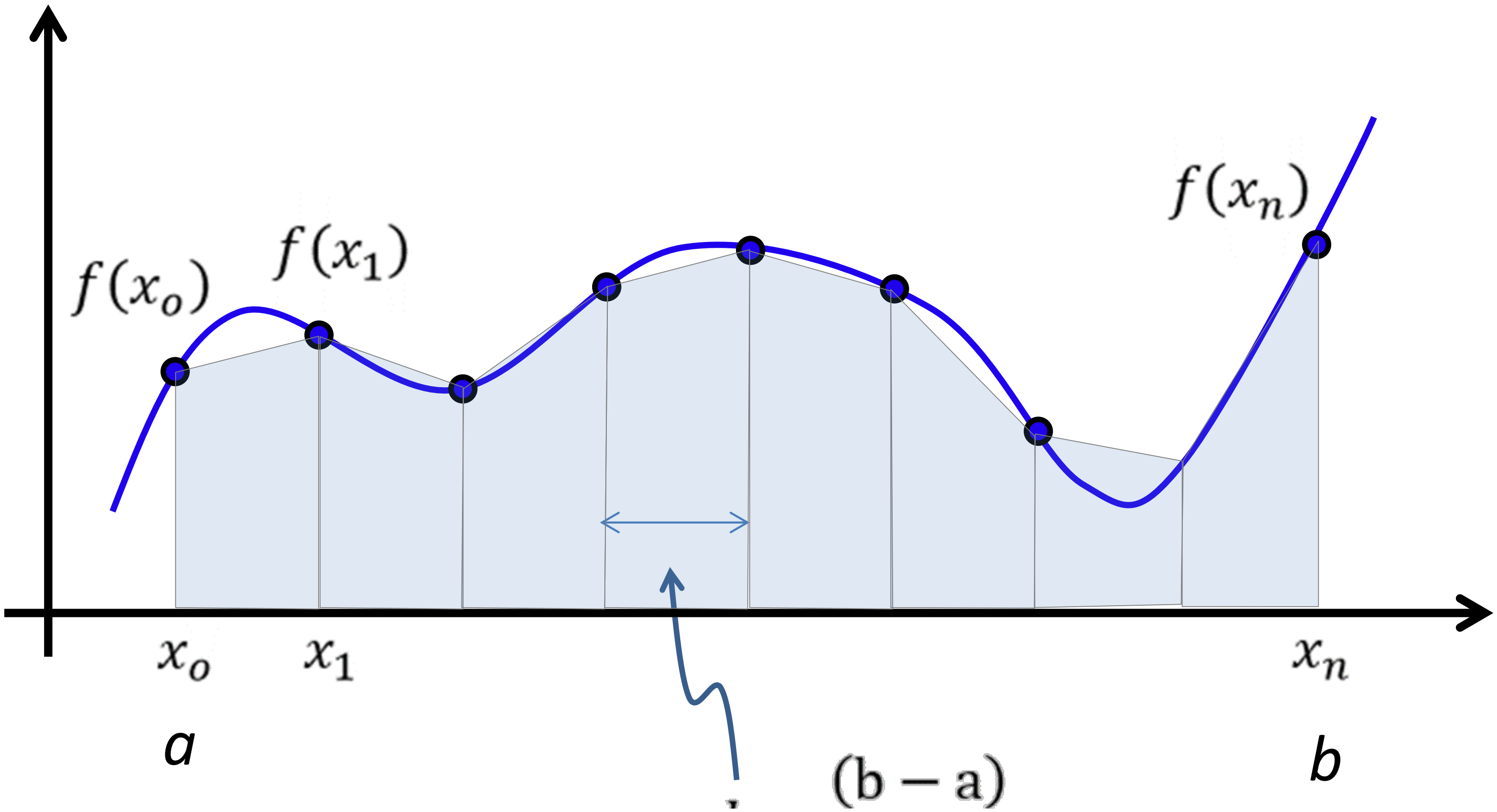


This suggests the error should be somewhere between -1.7 to 3.7.

If we looked at a much smaller interval (i.e., b-a << 0.7) we'd have a much more accurate answer.

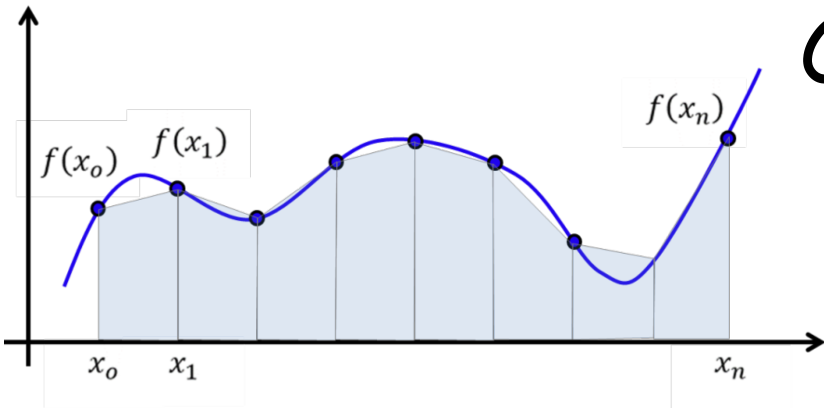
Question: what would the range of errors be if we were using a step size of 0.1 and were integrating from 0.2 to 0.3? What if we were integrating from 0.4 to 0.5, or 0.6 to 0.7?

Composite Trapezoidal Rule



$$\int_a^b f(x)dx = \int_{x_o}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx$$

This is exact! But to numerically solve this we could use the trapezoidal rule on each piece.



Composite Trapezoidal Rule

This is the exact solution:

$$\int_a^b f(x)dx = \int_{x_o}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx$$

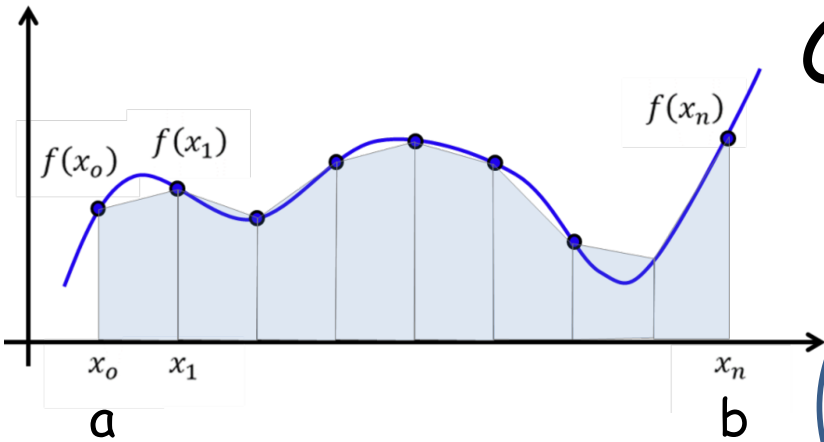
This is the composite trapezoidal rule:

$$\int_a^b f(x)dx \cong \frac{h}{2}[f(x_0) + f(x_1)] + \frac{h}{2}[f(x_1) + f(x_2)] + \cdots + \frac{h}{2}[f(x_{n-1}) + f(x_n)]$$

This is the composite trapezoidal rule stated a little more compactly (and easier to program, perhaps):

$$\int_a^b f(x)dx \cong \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

Composite Trapezoidal Rule



$$\int_a^b f(x)dx \cong \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

$$h = \frac{(b-a)}{n}$$

$$\int_a^b f(x)dx \cong (b-a)\frac{\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]}{2n}$$

width

This looks like the average height, with the end-points weighted only half as much as the inner points.

Error in the Composite Trapezoidal Rule

Recall the error when we used only one trapezoid:

$$E = -\frac{1}{12} f''(\xi)(\mathrm{b}-\mathrm{a})^3$$

If we divide the integral up into *n* pieces, we sum the individual errors:

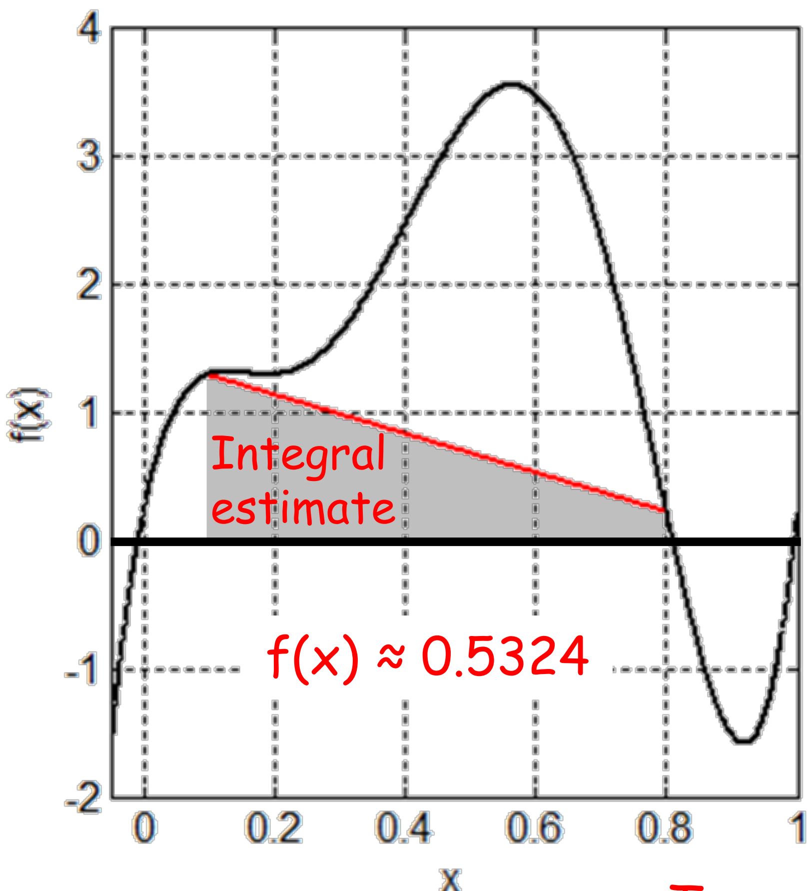
$$E = -\sum_{i=1}^{n} \frac{1}{12} f''(\xi_i) \left(\frac{\mathrm{b}-\mathrm{a}}{n}\right)^3$$

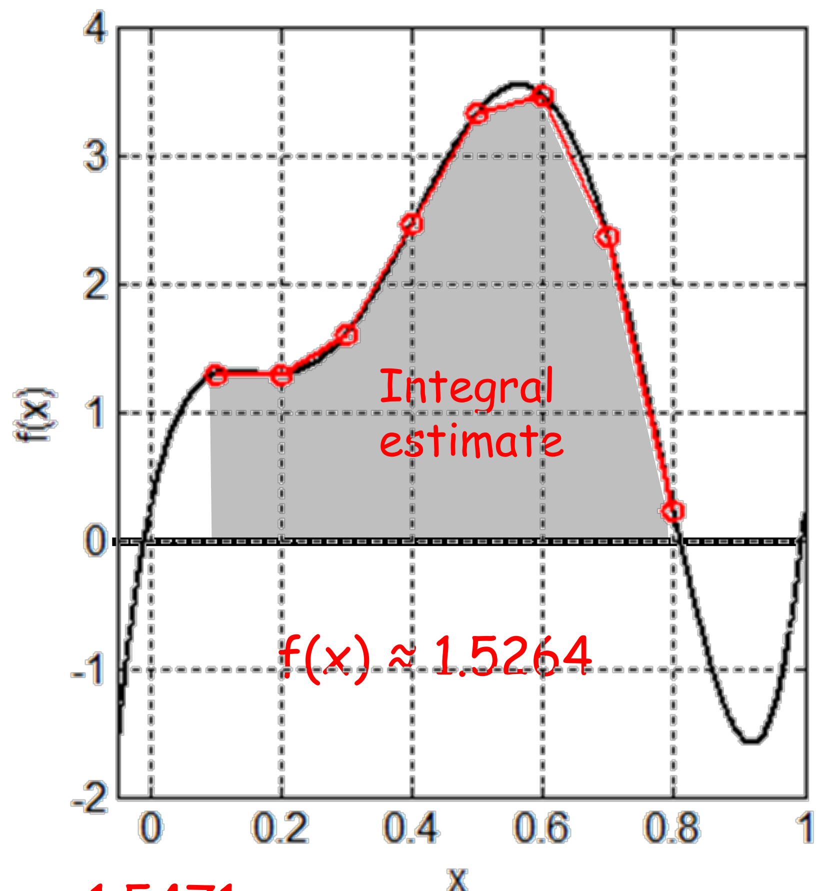
if this term is small, it helps!

Return to the previous example

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

Integrate *f(x)* from a = 0.1 to b=0.8



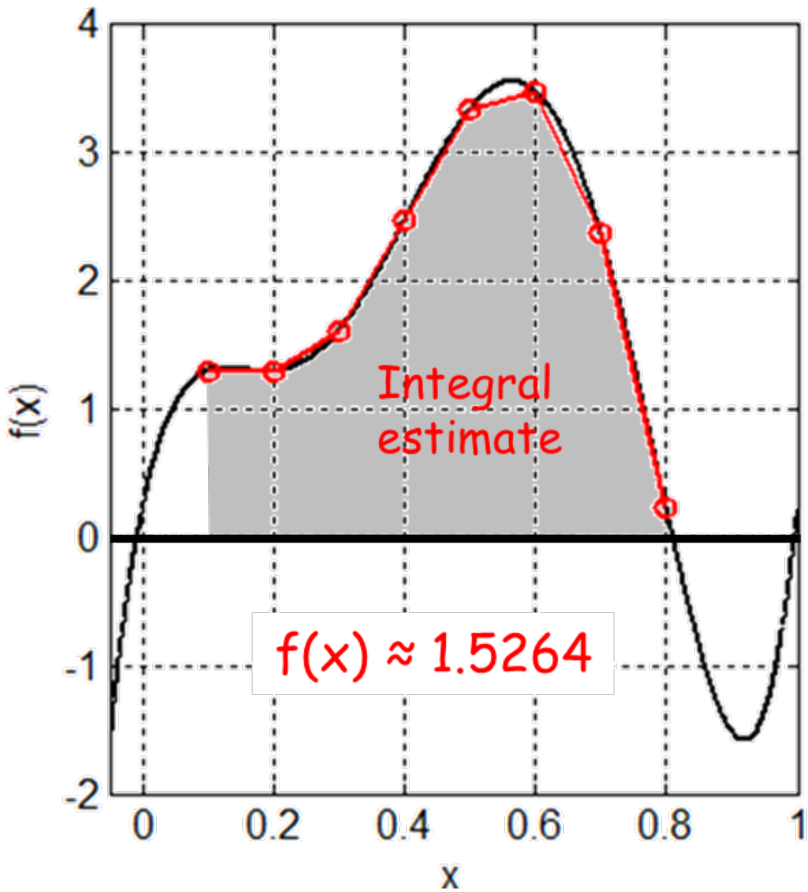


True answer: 1.5471

Return to the previous example

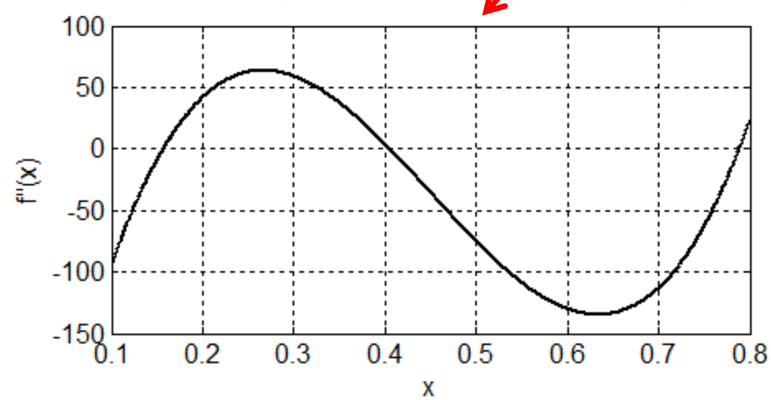
$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

Integrate $f(x)$ from a = 0.1 to b=0.8



True answer: 1.5471

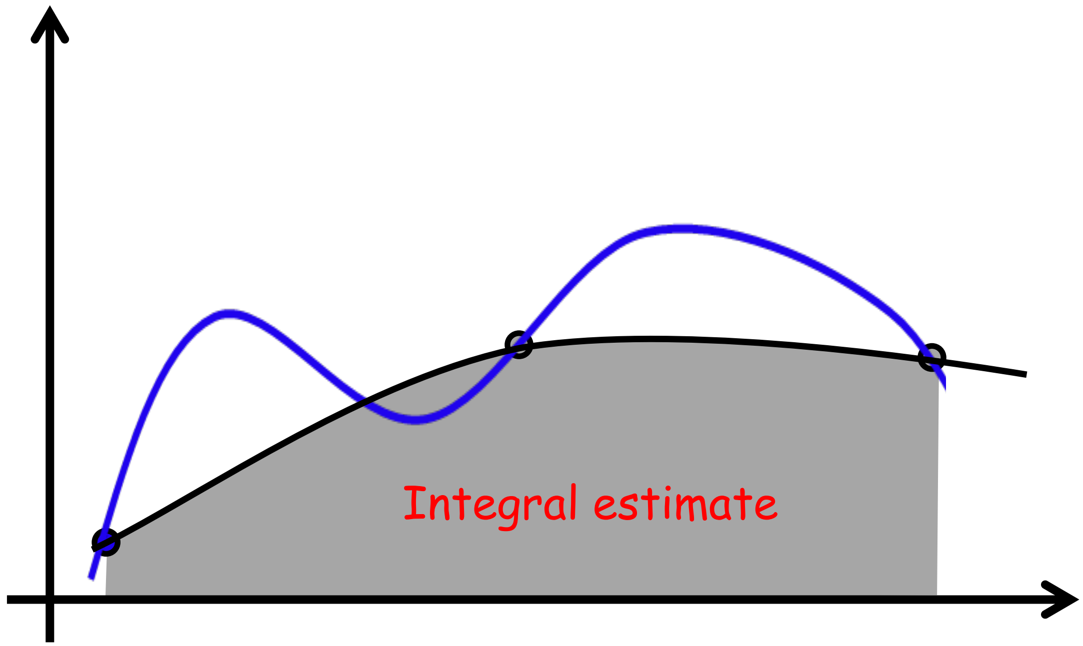
$$E = -\sum_{i=1}^{n} \frac{1}{12} f''(\xi_i) \left(\frac{b-a}{n}\right)^3$$



Error:
- f'' ~ 50
- (b-a)/n = 0.1
- 7 error terms to sum

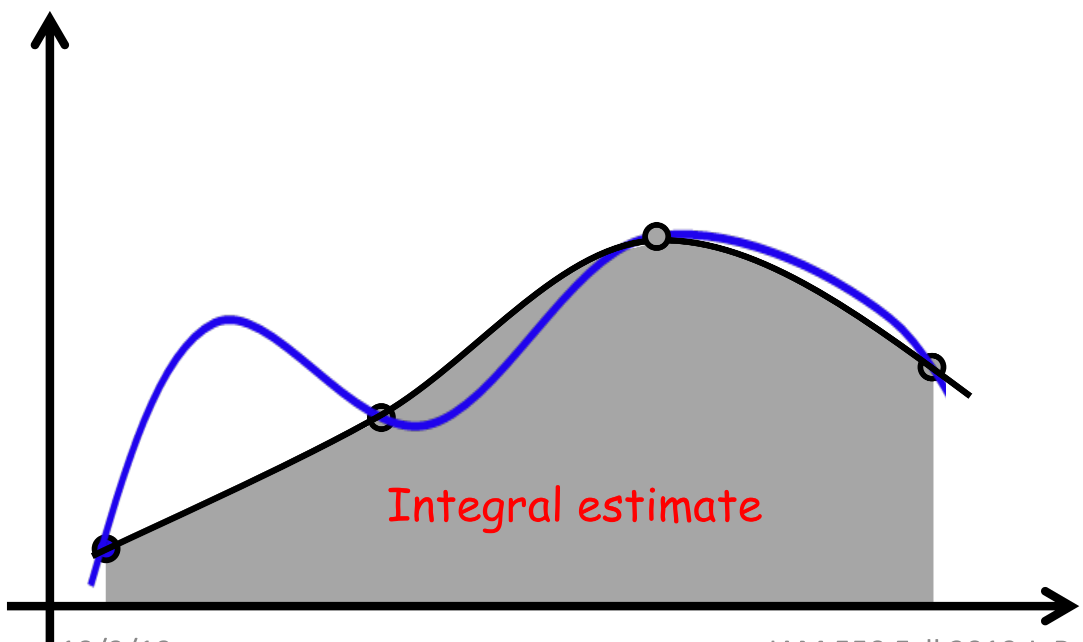
E ~ 0.03

Simpson's 1/3 Rule



If we add a midpoint to the trapezoidal rule, we can fit a higher-order polynomial.

3 points – Parabola
Gives us Simpsons 1/3 Rule



4 points – 3rd order polynomial
Gives us Simpsons 3/8 Rule

Recall Lagrange's Approximation

$$P_N(x) = \sum_{k=0}^{N} y_k L_{N,k}(x)$$

For Simpsons 1/3 rule, there will be 3 terms in this summation.

$$L_{N,k}(x) = \frac{(x - x_o)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_N)}{(x_k - x_o)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_N)}$$

N is the order of the polynomial

Note that $(x\text{-}x_k)$ and $(x_k\text{-}x_k)$ are not present here.

After we figure out the correct form for Lagrange's approximation, we'll integrate it.

Recall Lagrange's Approximation

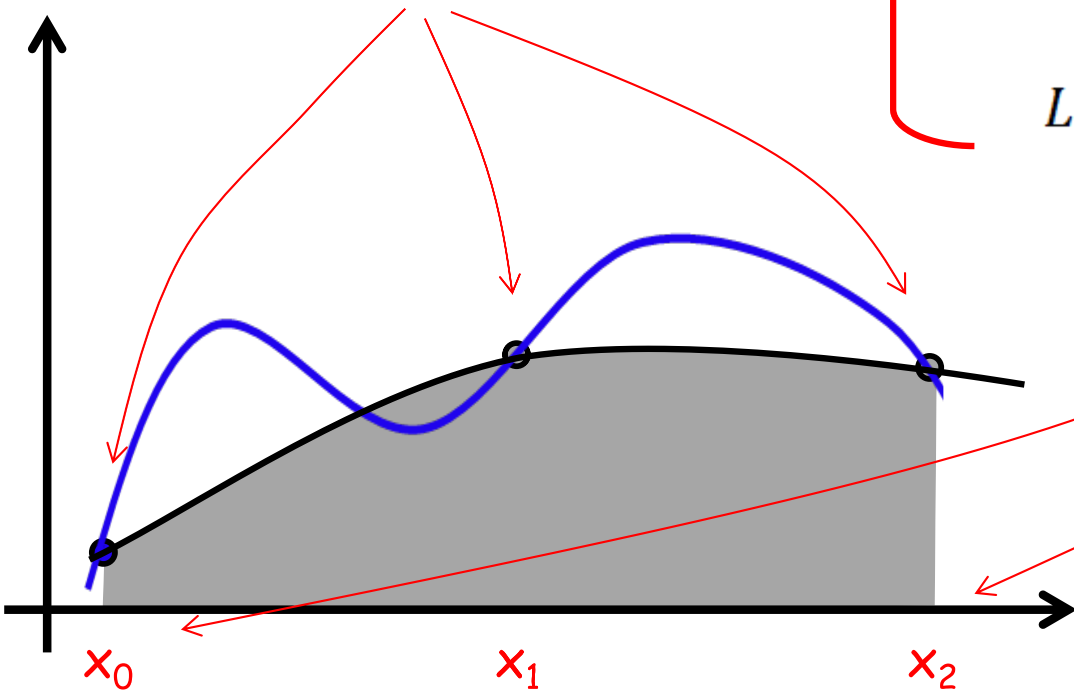
$$P_N(x) = \sum_{k=0}^{N} y_k L_{N,k}(x)$$

$$L_{2,0}(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

$$L_{2,2}(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

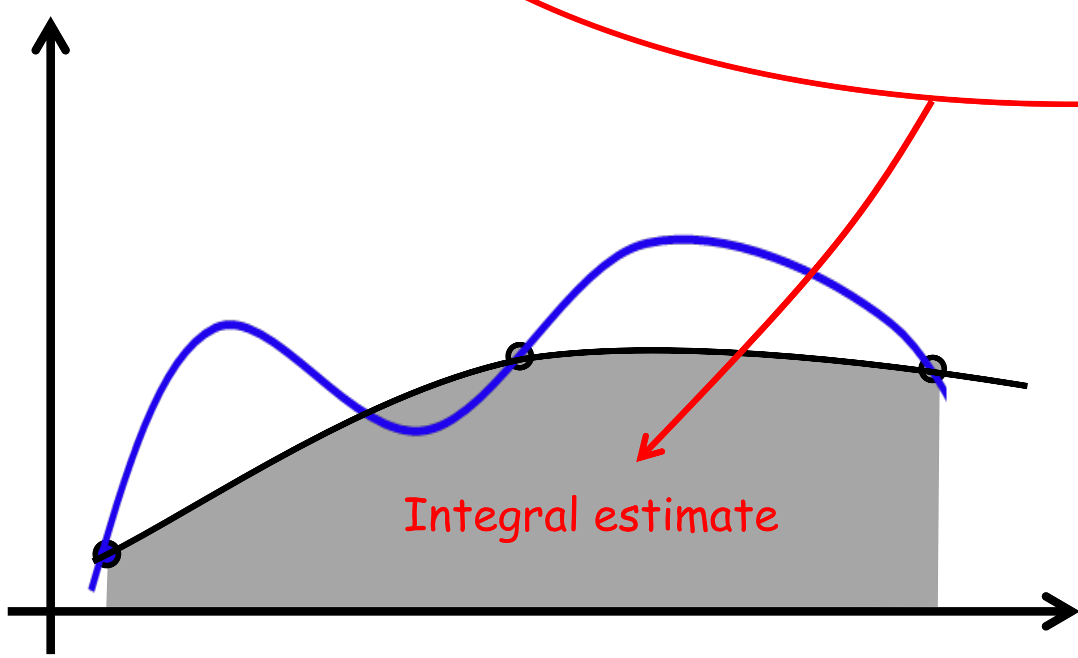
$$L_{2,2}(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

a.k.a. $f(x_k)$



Simpson's 1/3 Rule

$$\int_{x_0}^{x_2} f(x)dx \cong \int_{x_0}^{x_2} \left[f(x_o)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \right] dx$$

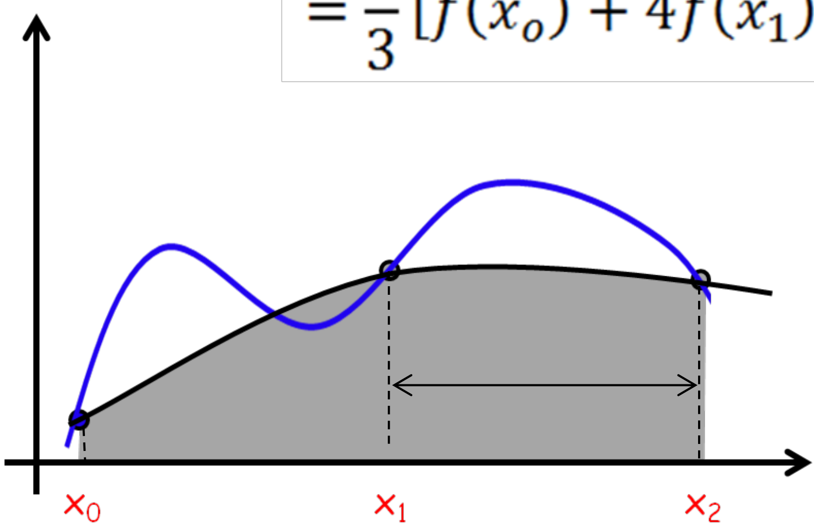


Simpson's 1/3 Rule

$$\int_{x_0}^{x_2} f(x)dx \cong \int_{x_0}^{x_2} \left[f(x_o)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \right] dx$$

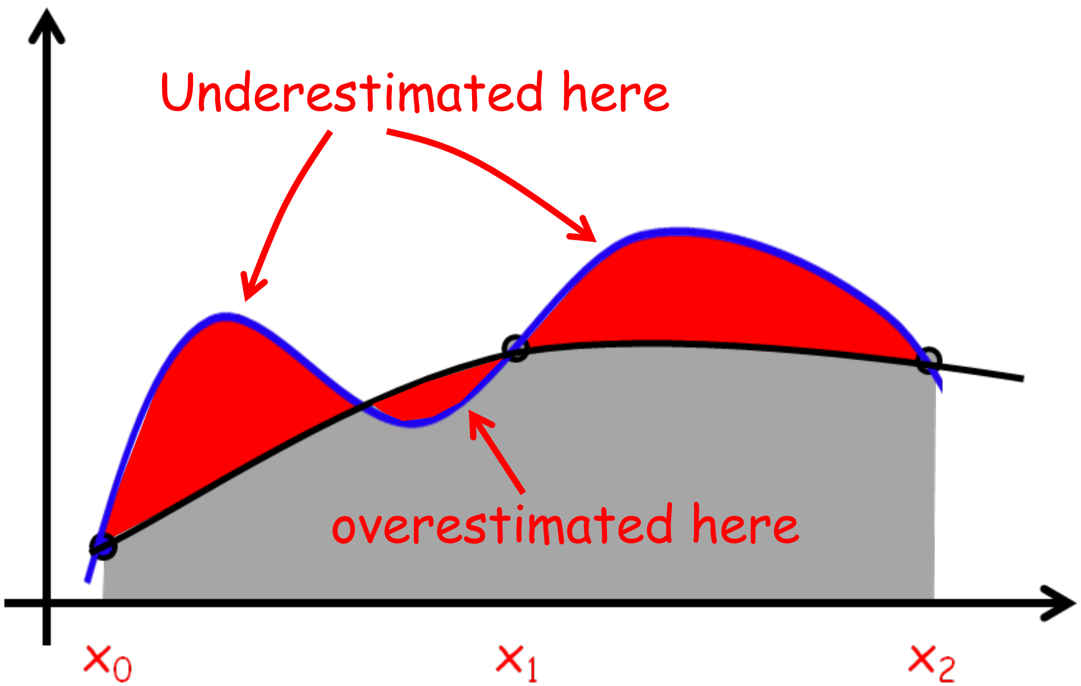
$$= \frac{h}{3}[f(x_o) + 4f(x_1) + f(x_2)]$$

Note the 1/3 → rule name



$$h = \frac{b-a}{2} = \frac{x_2 - x_0}{2}$$

Error in Simpson's 1/3 Rule



$$E = -\frac{1}{90} f^{(4)}(\xi) h^5$$

$$E = -\frac{1}{2880} f^{(4)}(\xi)(b-a)^5$$

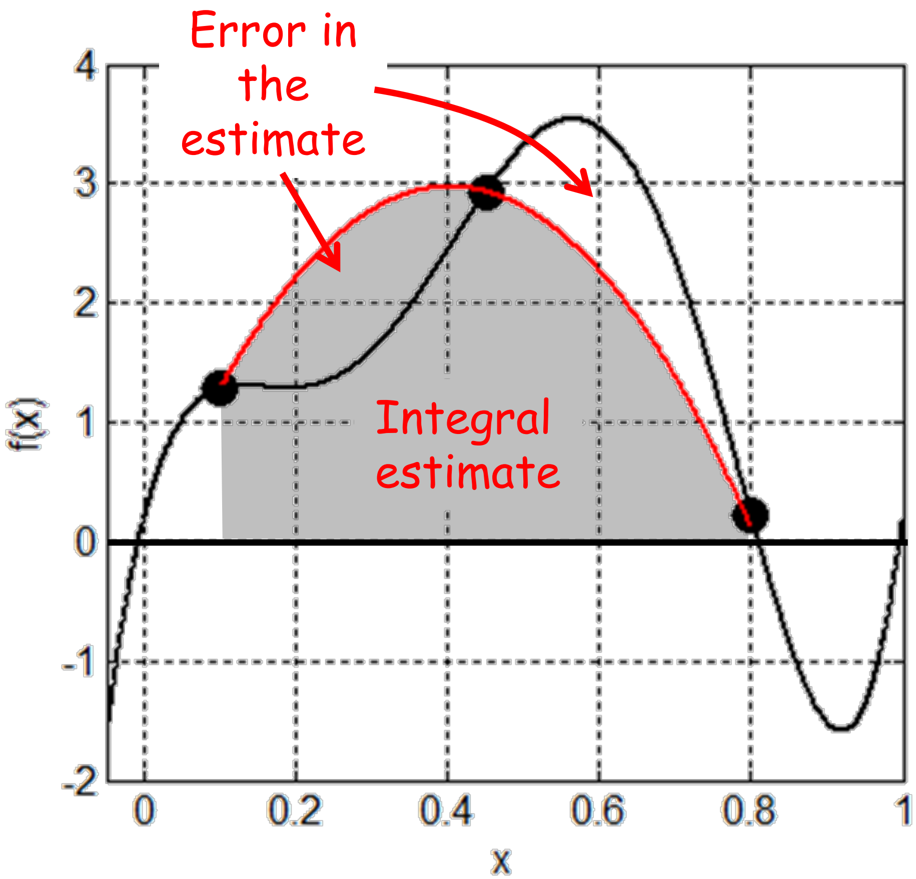
The error is zero for cubic polynomials!!

Caution: as before when developed the trapezoidal rule, this error only applies to single-segment applications of Simpson's 1/3 Rule

Example

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

Integrate *f(x)* from a = 0.1 to b=0.8



True answer: 1.5471

f(0.1) = 1.289
f(0.45) = 2.935
f(0.8) = 0.232
h = 0.35

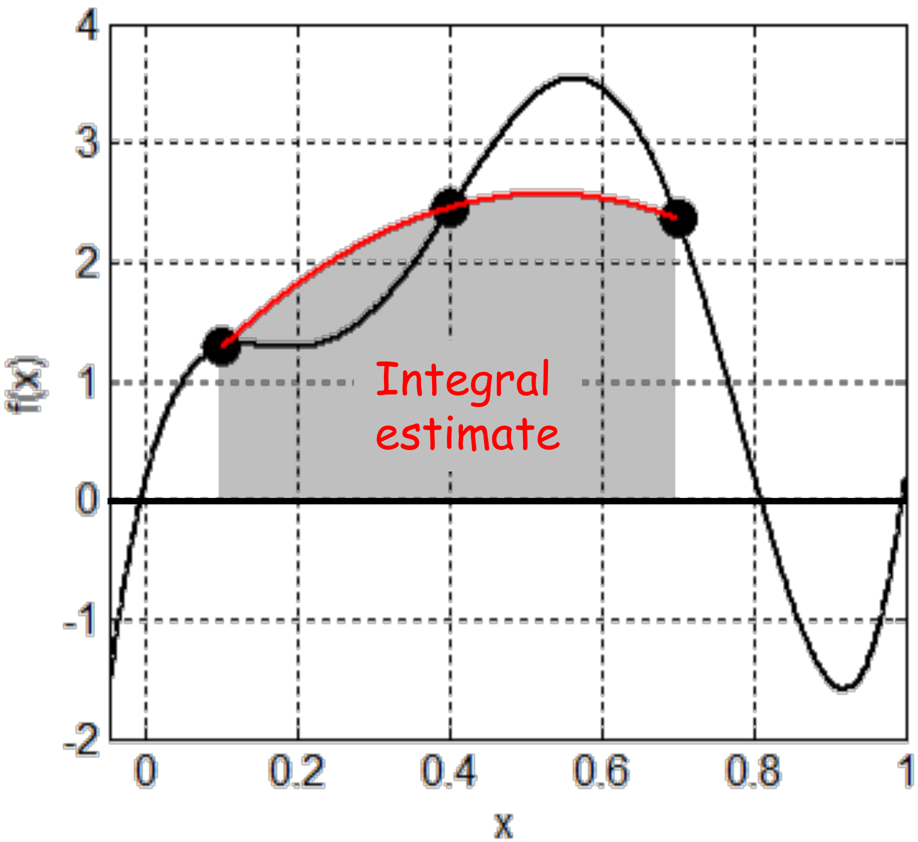
Simpsons 1/3 rule estimate:
h/3*[f(0.1) + 4*f(0.45) +f(0.8)] =
1.5471

E = 0 ???

Example

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

Integrate *f(x)* from a = 0.1 to b=**0.7**



True answer: 1.4124

f(0.1) = 1.289
f(**0.4**) = 2.456
f(**0.7**) = 2.363
h = 0.3

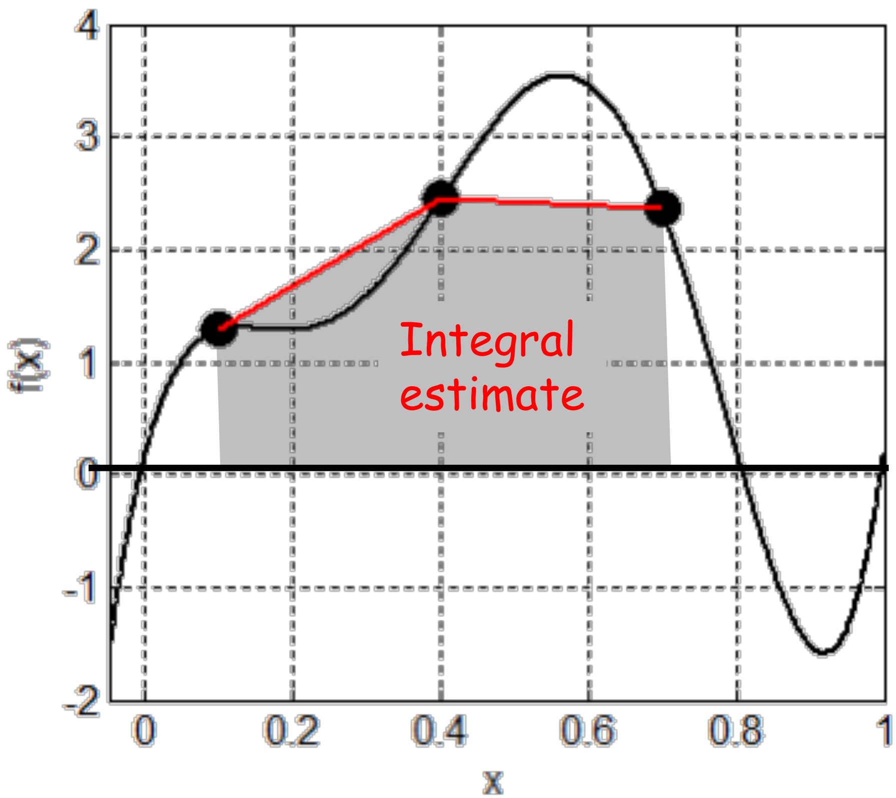
Simpsons 1/3 rule estimate:
h/3*[f(0.1) + 4*f(**0.4**) +f(**0.7**)] =
1.348

E = 0.064
%E = 4.6%

Example

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

Integrate *f(x)* from a = 0.1 to b=**0.7**
but use the trapezoidal rule for comparison



True answer: 1.4124

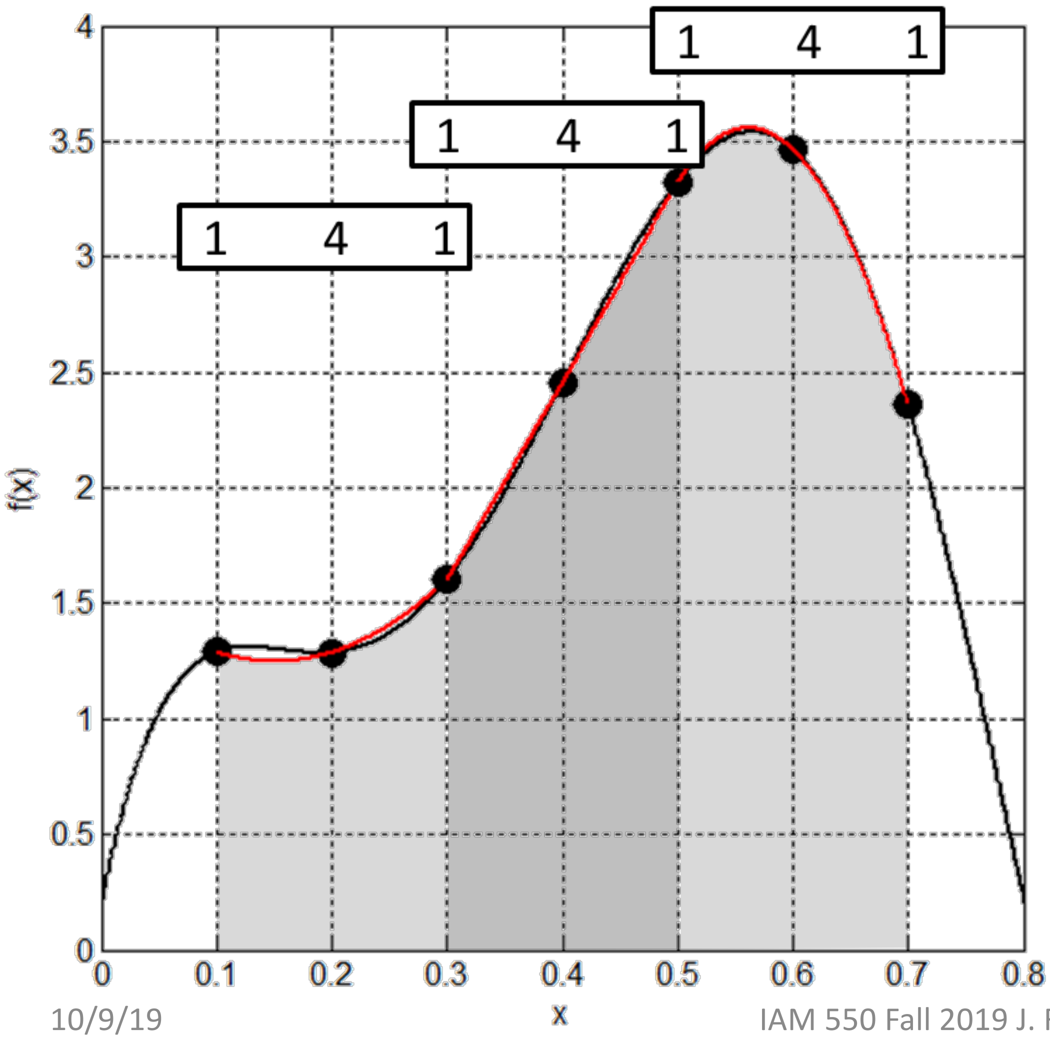
f(0.1) = 1.289
f(0.4) = 2.456
f(0.7) = 2.363
h = 0.3

Trapezoidal rule estimate:
h/2*[f(0.1) + 2*f(0.4) +f(0.7)] = 1.285

E = 0.127
%E = 9.0%

Composite Simpson's 1/3 Rule

$$\int_a^b f(x)dx = \int_{x_o}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)dx$$



Notes

1. we need an odd number of points (x_i) to be able to do this
2. The 'even' points are used twice as much as the 'odd' points

Composite Simpson's 1/3 Rule

$$\int_a^b f(x)dx = \int_{x_o}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)dx$$

$$\int_a^b f(x)dx \cong 2h\frac{[f(x_o) + 4f(x_1) + f(x_2)]}{6} + 2h\frac{[f(x_2) + 4f(x_3) + f(x_4)]}{6} + \cdots$$

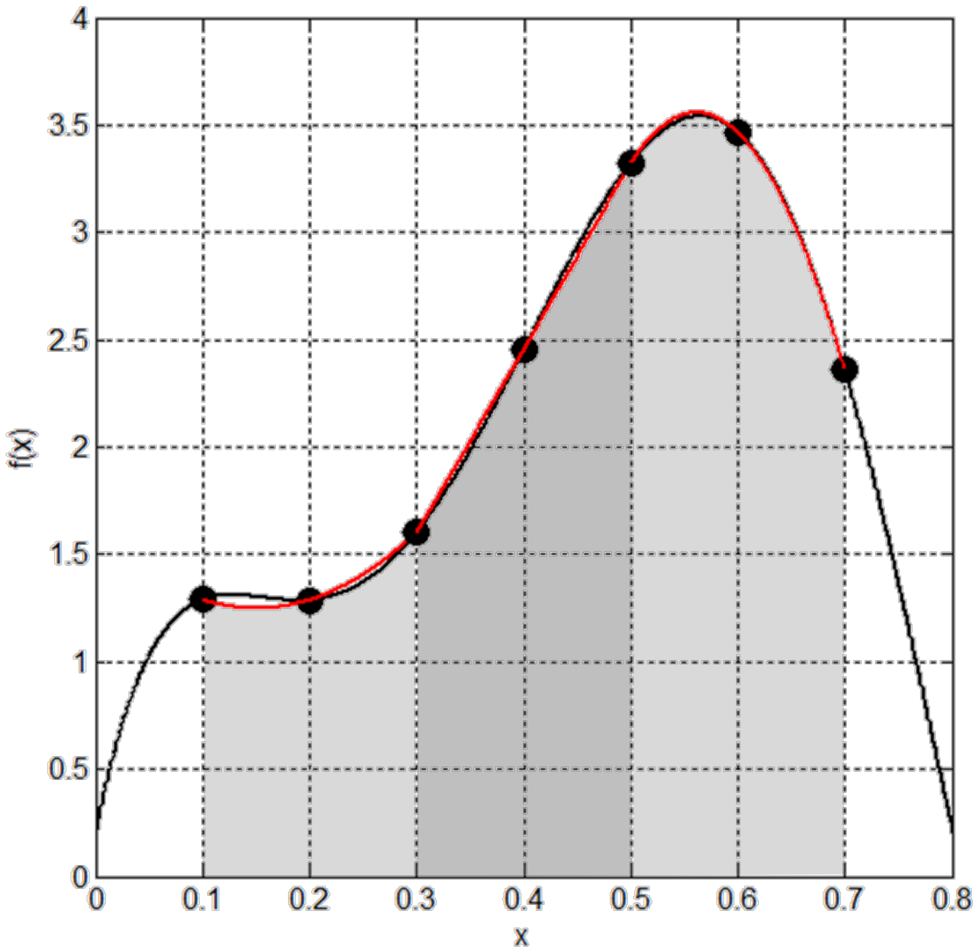
$$+ 2h\frac{[f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]}{6}$$

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{i=2,4,6}^{n-2} f(x_i) + f(x_n)\right]$$

Return to Our Example

True answer: 1.4124

Integrate *f(x)* from a = 0.1 to b=**0.7**



f(0.1) = 1.289 } 1 of these

f(0.2) = 1.288
f(0.4) = 2.456
f(0.6) = 3.464 } 4 of these

f(0.3) = 1.607
f(0.5) = 3.325 } 2 of these

f(0.7) = 2.363 } 1 of these

h = 0.1

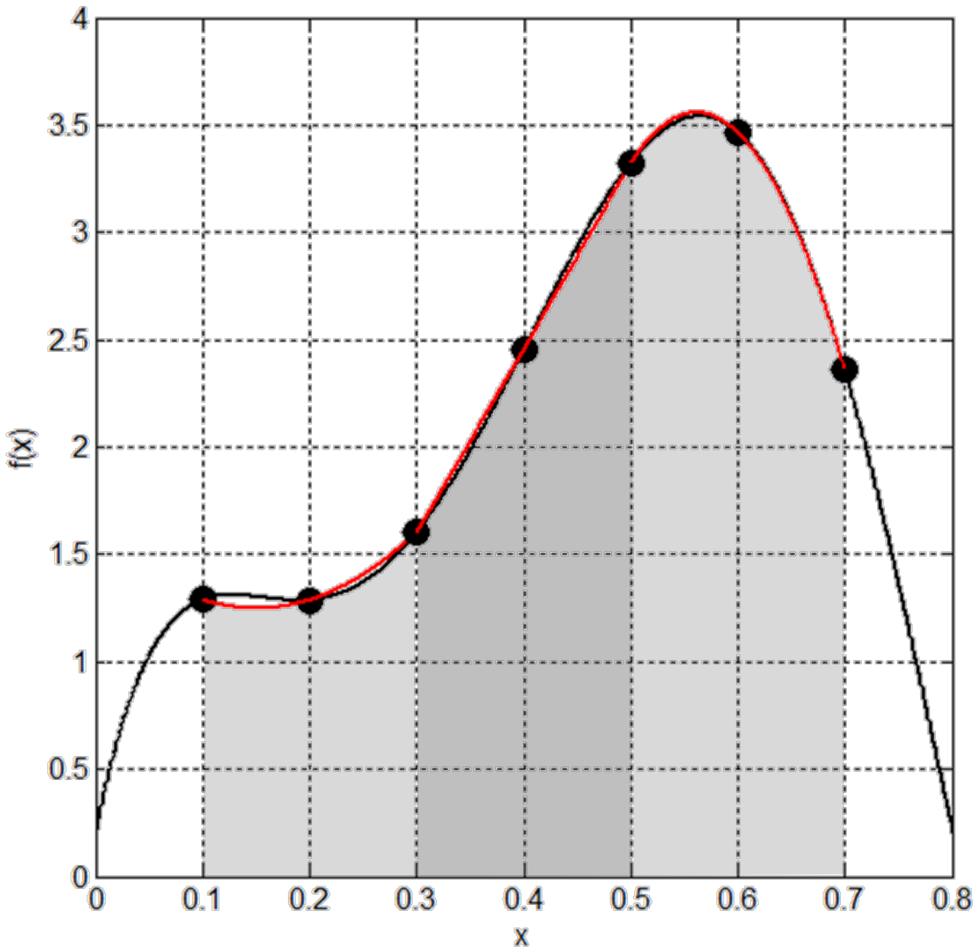
Composite Simpsons 1/3 rule estimate:
1.4116

E = 0.0008 %E = 0.06%

Return to Our Example

True answer: 1.4124

Integrate *f(x)* from a = 0.1 to b=**0.7**



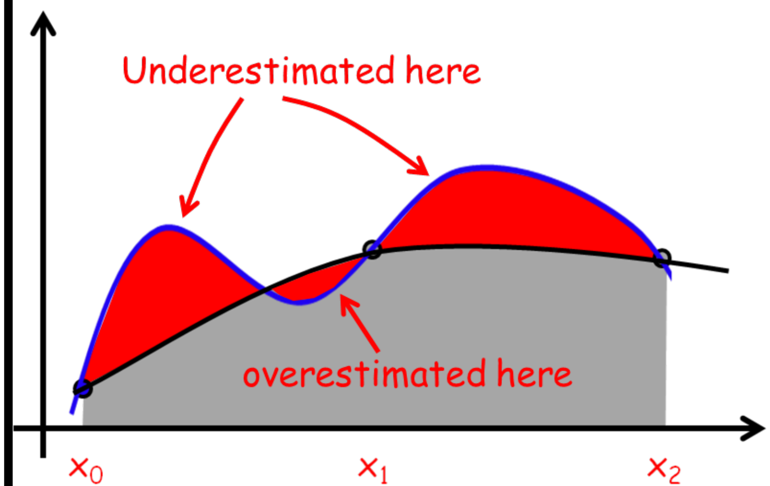
Composite Simpsons 1/3 rule estimate:
1.4116

E = 0.0008 %E = 0.06%

Trapezoidal Rule estimate with same points:
1.3966
E = 0.0158 %E = 1.12%

Note! If we wanted to integrate to from 0.1 to 0.8, we can't use steps of 0.1 with Simpsons 1/3 rule.

Error in the Composite Simpson's 1/3 Rule



Single Segment

$$E = -\frac{1}{90} f^{(4)}(\xi) \mathrm{h}^5$$

$$E = -\frac{1}{2880} f^{(4)}(\xi) (\mathrm{b}-\mathrm{a})^5$$

If we divide the integral up into *m* segments, we sum the individual errors:

$$E = -\sum_{i=1}^{m} \frac{1}{2880} f^{(4)}(\xi_i) \left(\frac{\mathrm{b}-\mathrm{a}}{2m}\right)^5$$

If there are n points, then there are m = (n-1)/2 segments

Newton-Cotes Integration Formulas

We've done 1st order Lagrange polynomials (trapezoidal rule) and 2nd order Lagrange polynomials (Simpsons 1/3 rule), and can keep going.

From Chapra, Applied Numerical Methods with MATLAB, 3rd Ed. p. 481

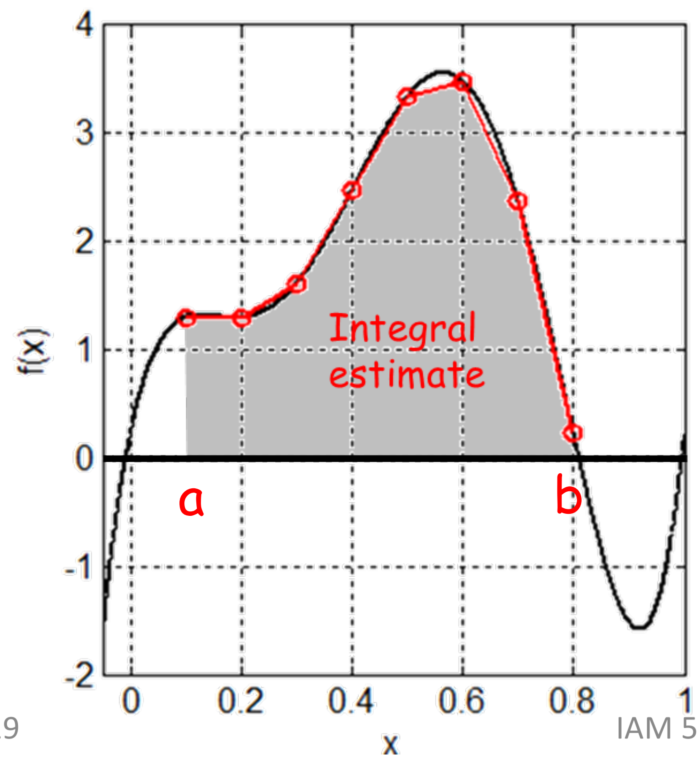
TABLE 19.2 Newton-Cotes closed integration formulas. The formulas are presented in the format of Eq. (19.13) so that the weighting of the data points to estimate the average height is apparent. The step size is given by $h = (b - a)/n$.

Segments (n)	Points	Name	Formula	Truncation Error
1	2	Trapezoidal rule	$(b-a)\dfrac{f(x_0)+f(x_1)}{2}$	$-(1/12)h^3 f''(\xi)$
2	3	Simpson's 1/3 rule	$(b-a)\dfrac{f(x_0)+4f(x_1)+f(x_2)}{6}$	$-(1/90)h^5 f^{(4)}(\xi)$
3	4	Simpson's 3/8 rule	$(b-a)\dfrac{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)}{8}$	$-(3/80)h^5 f^{(4)}(\xi)$
4	5	Boole's rule	$(b-a)\dfrac{7f(x_0)+32f(x_1)+12f(x_2)+32f(x_3)+7f(x_4)}{90}$	$-(8/945)h^7 f^{(6)}(\xi)$
5	6		$(b-a)\dfrac{19f(x_0)+75f(x_1)+50f(x_2)+50f(x_3)+75f(x_4)+19f(x_5)}{288}$	$-(275/12{,}096)h^7 f^{(6)}(\xi)$

Romberg Integration

$$I = \hat{I} + E$$

$$E = -\sum_{i=1}^{n} \frac{1}{12} f''(\xi_i) \left(\frac{\mathrm{b-a}}{n}\right)^3 \approx -\frac{(\mathrm{b-a})^3}{12n^2}\overline{f''} = \frac{(\mathrm{b-a})}{12} h^2 \overline{f''}$$

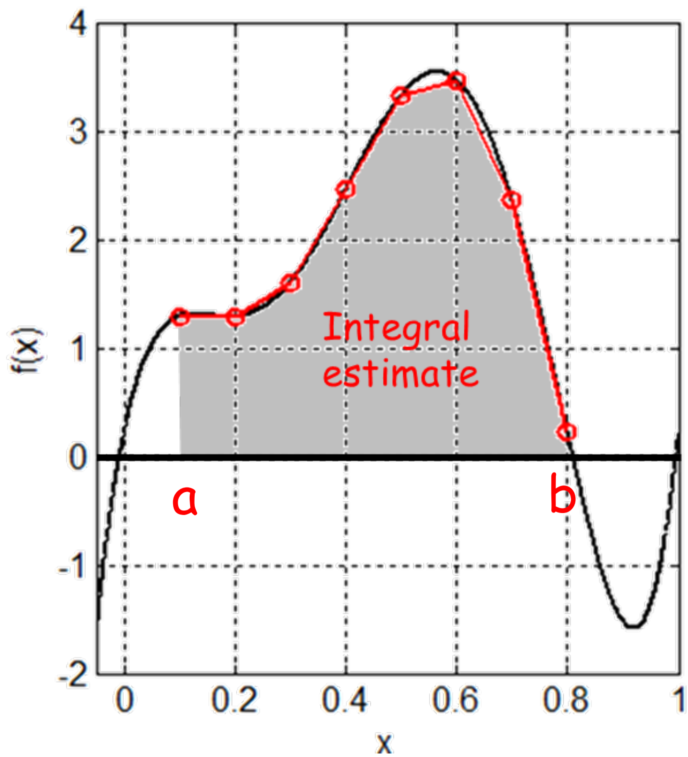


n=7
b-a = 0.7
h = 0.1

Note: b-a = n*h

Romberg Integration

What if I made two separate estimates, with different step sizes (recall Richardson extrapolation)?



Estimate 1, with step size h1:

$$I = \hat{I}(h_1) + E(h_1)$$

Estimate 2, with step size h2:

$$I = \hat{I}(h_2) + E(h_2)$$

$$\hat{I}(h_1) + E(h_1) = \hat{I}(h_2) + E(h_2)$$

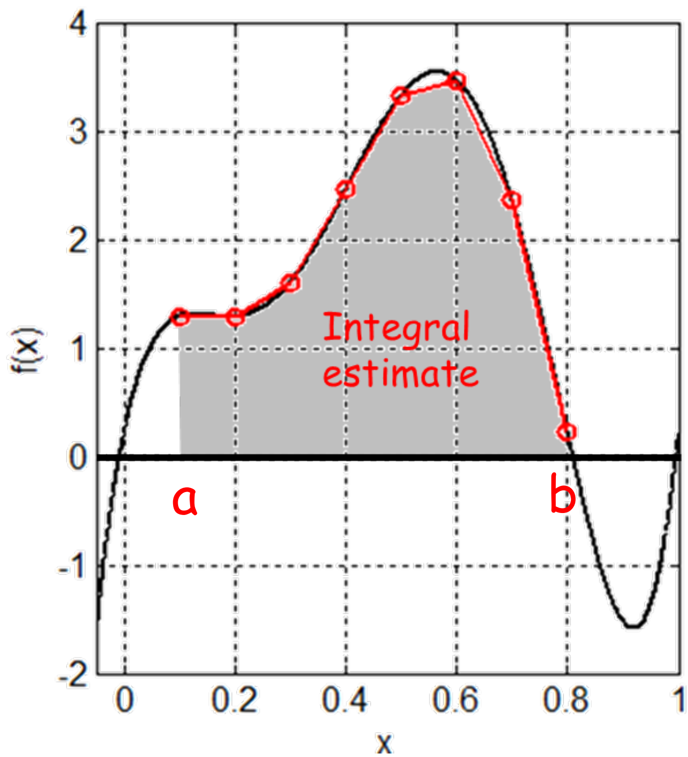
Each estimate will have an error that depends similarly on the 2nd derivative but with different step sizes:

$$E_1 = \frac{(b-a)}{12} h_1^2 \overline{f''} \qquad E_2 = \frac{(b-a)}{12} h_2^2 \overline{f''}$$

$$\frac{E_1}{E_2} \cong \frac{h_1^2}{h_2^2}$$

Romberg Integration

What if I made two separate estimates, with different step sizes (known as Richardson extrapolation)



Solve for E2 and substitute:

$$\frac{E_1}{E_2} \cong \frac{h_1^2}{h_2^2}$$

$$\hat{I}(h_1) + E(h_2)\frac{h_1^2}{h_2^2} = \hat{I}(h_2) + E(h_2)$$

Solve for E2

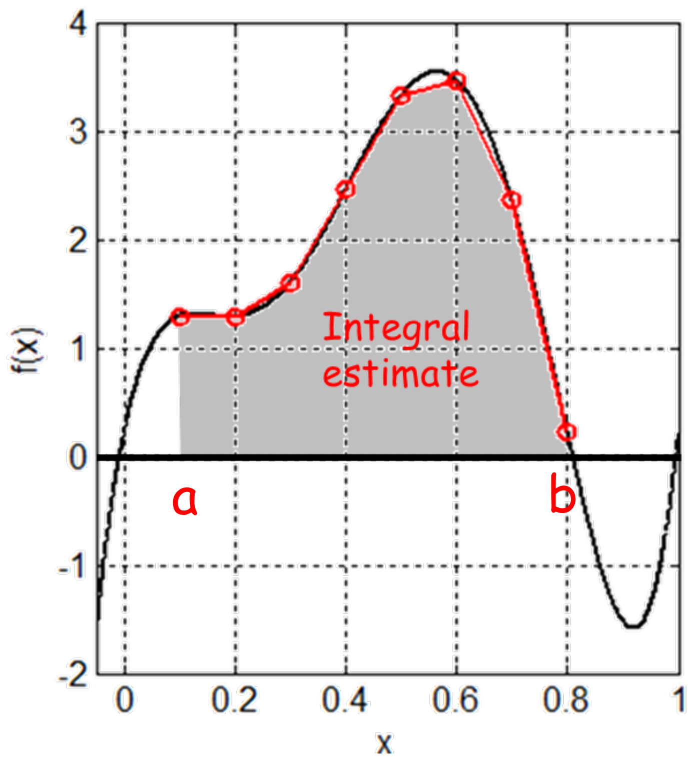
$$E(h_2) = \frac{\hat{I}(h_1) - \hat{I}(h_2)}{\left(1 - \frac{h_1^2}{h_2^2}\right)}$$

This should be an 'approximately equal'

We know (or can calculate) all of these terms!

Romberg Integration

What if I made two separate estimates, with different step sizes (Richardson extrapolation)



We can now use our estimate of the *error* to improve our estimate of the integral:

$$I = \hat{I}(h_2) + E(h_2) = \hat{I}(h_2) + \frac{\hat{I}(h_1) - \hat{I}(h_2)}{\left(1 - \frac{h_1^2}{h_2^2}\right)}$$

$$E(h_2) = \frac{\hat{I}(h_1) - \hat{I}(h_2)}{\left(1 - \frac{h_1^2}{h_2^2}\right)}$$

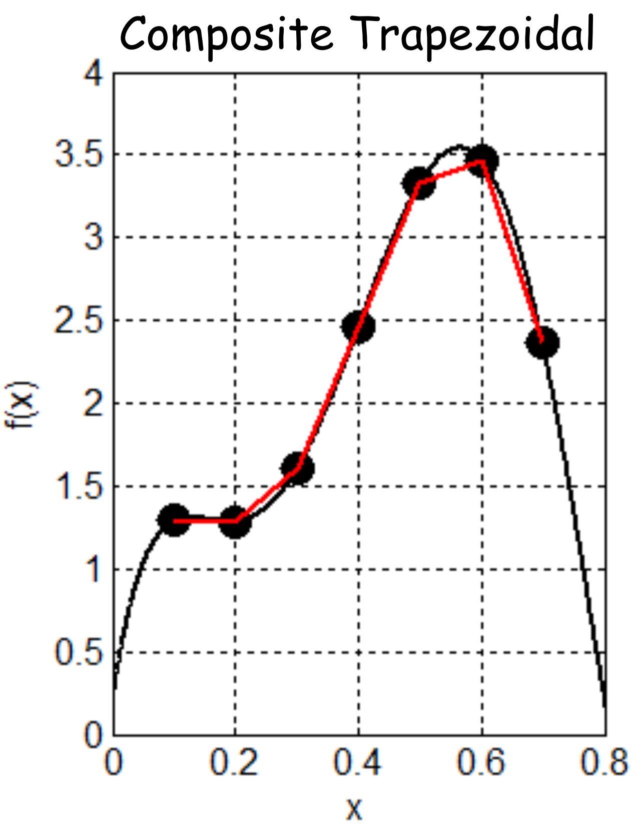
If h2 = h1/2:

$$I = \frac{4}{3}\hat{I}(h_1) - \frac{1}{3}\hat{I}(h_2)$$

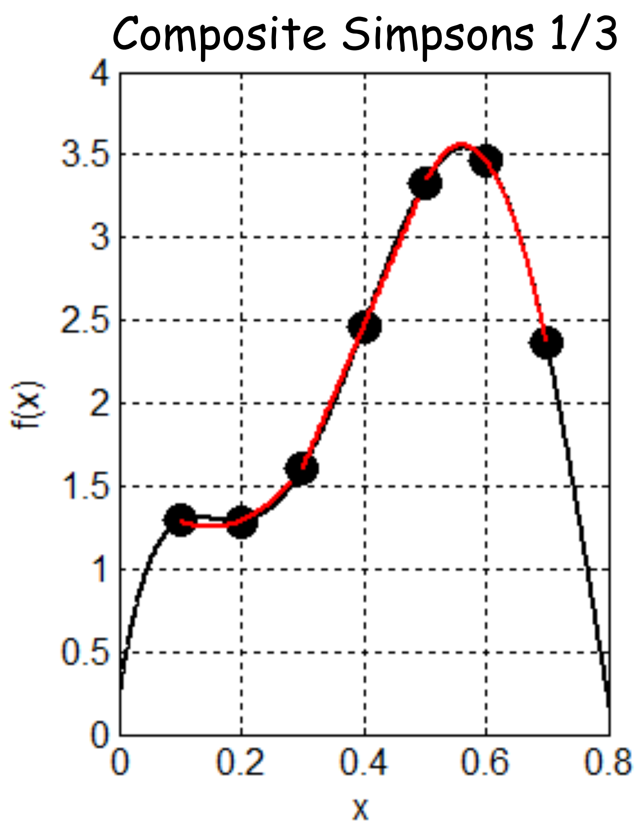
This turns out to be $O(h^4)$.

With h2=h1/2 you can reuse the $f(x_i)$ from I_2 to save computation time.

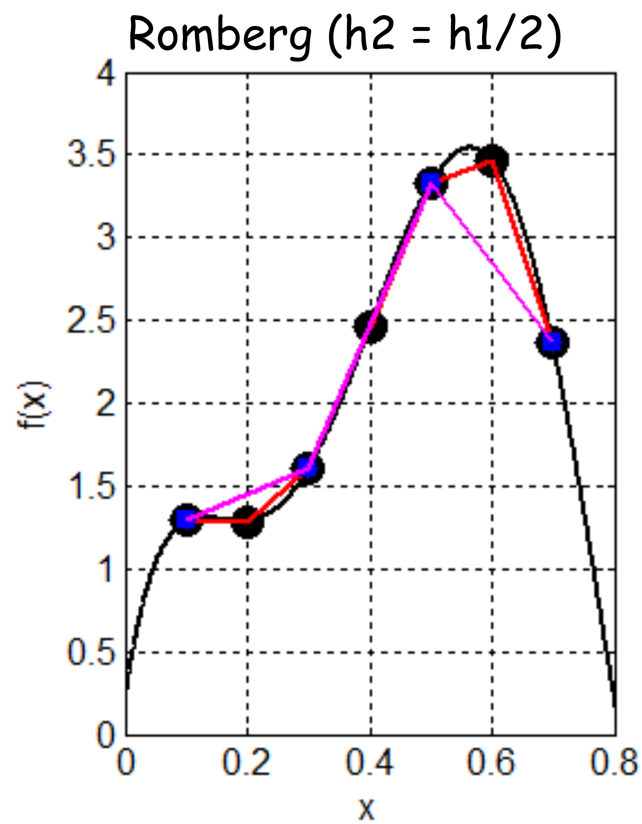
Comparison: Trapezoidal, Simpsons 1/3, Romberg



Integral Result: 1.3966



Integral Result: 1.4116



Integral Result: 1.41116

True Answer: 1.4124

The Typical Romberg Integration Scheme Uses Iterations:

Trap for step size: (b-a)
Trap for step size: (b-a)/2
→ Result of iteration 1

Trap for step size: (b-a)
Trap for step size: (b-a)/2
Trap for step size: (b-a)/4
→ Result of iteration 1
→ Intermediate iteration 2
→ Result of iteration 2

Trap for step size: (b-a)
Trap for step size: (b-a)/2
Trap for step size: (b-a)/4
Trap for step size: (b-a)/8
→ Result of iteration 1
→ Intermediate iteration 2
→ Intermediate iteration 3
→ Result of iteration 2
→ Intermediate iteration 3
→ Ans.

Overkill for us, but MATLAB does something like this with the integrate() function. Romberg has the advantage that it provides an error estimate without knowing the true integral. You can specify a precision and stop the iterations when it is reached.

Take-home messages

- We have several methods for doing numerical integration
  - Left/right-end rule
  - Trapezoidal Rule
  - Simpsons 1/3 Rule
  - Romberg
  - Other extended methods (lots of them, Gauss-Legendre)
- In all cases: small step size helps!
- If you have f() given by data, trapezoidal is good enough, because your data have errors to begin with.
- You should spend time with these notes, working through the examples and making sure you understand the different methods.